1.1-1.2 Vector Algebra Lecture

Saturday, January 23, 2021 7:02 PM

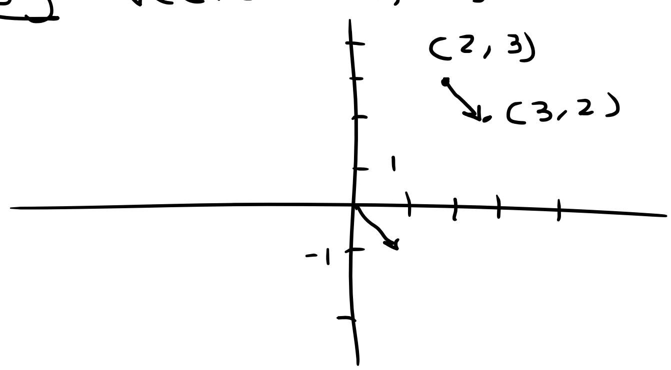
plane: set of pairs (a,b) or (x,y) of real numbers (in Cartesian coord system)
"Euclidean plane" or "$\mathbb{R}^2$"

Euclidean Space: set of triples of real #'s or $\mathbb{R}^3$

Vector: basically just a point in the plane (plane vectors) or space (space vectors)

We think of it as a directed line segment from $(0,0)$ [or $(0,0,0)$] to that point

eg Vector $(1,-1)$ looks like:



so the vector $(-1,1)$ aka vector from $(0,0)$ to $(1,-1)$ is the same as the vector from $(2,3)$ to $(3,2)$

generally vector from $P=(x_0,y_0)$ to $Q=(x_1,y_1)$ is $\vec{v}=(x_1-x_0, y_1-y_0)$

Vector Terminology
(in terms of space vectors)

space v. plane?

Vector Terminology

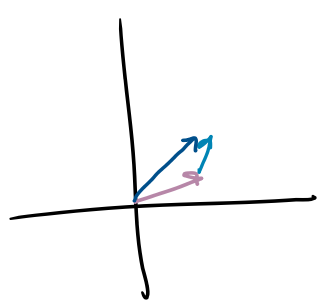
(in terms of space vectors)

space v. plane?

Let $\vec{v} = (x, y, z)$

- Coordinates are x, y & z
  aka: x-coord, y-coord, z-coord
  OR i- j- k-
- Coordinate vectors
  $\vec{i} = (1, 0, 0)$
  $\vec{k} = (0, 0, 1)$
  $\vec{j} = (0, 1, 0)$

- vectors are equal iff their coordinates are all equal
- Can add vectors
  - alg: add coords
  - geo: to add $\vec{v}$ and $\vec{w}$, put base of $\vec{w}$ at endpoint of $\vec{v}$ where and see where endpoint of $\vec{w}$ lands
    so $\vec{v} + \vec{w}$ = vector from base $\vec{v}$ to endpoint $\vec{w}$



- Can multiply vectors by scalar

- satisfy commutative, associative, distributive, etc (see Theorem 1.5)

associative, distributive, etc
(see Theorem 1.5)

- $\vec{v}$ and $\vec{w}$ are parallel if one is scalar mult of other

- $\vec{v}$ and $\vec{w}$ are in same direction iff one is a positive scalar multiple of the other

- Magnitude $||\vec{v}||$
$= \sqrt{\text{sum of squares of coords}}$
aka length of vector by Pythagorean Theorem

$||\vec{n}||$
denotes magnitude

$||\vec{PQ}||$
magnitude from P to Q

- For vector $\vec{v}$ and scalar a,
$||a \cdot \vec{v}|| = |a| \cdot ||\vec{v}||.$

- unit vector has magnitude 1.

If $\vec{v}$ is nonzero, then:
$\dfrac{\vec{v}}{||\vec{v}||}$ is a unit vector in the same direction as $\vec{v}$.

Up Next: DOT PRODUCTS

Dot products

Recall $||\vec{PQ}||$ = distance from P to Q

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Sqrts are hard, linear fcns are easier

Fact $||v||^2 = v \cdot v$

↑ dot product

can use dot prod to understand distance

Dot product is bilinear — linear in each vector

ie. $(v_1 + v_2) \cdot w = v_1 \cdot w + v_2 \cdot w$

vectors

If k is a real # then:

$(kv) \cdot w = k(v \cdot w)$.

Similarly, $v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2$

$v \cdot (kw) = k(v \cdot w) = (kv) \cdot w$

eg. $||v+w||^2 = (v+w) \cdot (v+w)$

$= v \cdot (v+w) + w \cdot (v+w)$

$= v \cdot v + v \cdot w + w \cdot v + w \cdot w$

$= ||v||^2 + ||w||^2 + 2 v \cdot w$

For $\underset{\text{vector}}{\vec{v}} = (\underbrace{v_1, v_2, v_3}_{\substack{\text{coords} \\ \text{(scalars)}}})$ and $\vec{w} = (w_1, w_2, w_3)$

then

$\vec{v} \cdot \vec{w} := v_1 w_1 + v_2 w_2 + v_3 w_3$

(dot product in 3-dimensions)

In n dimensions

$\vec{v} = (v_1, \ldots, v_n)$

$\vec{w} = (w_1, \ldots, w_n)$

then

$$\vec{v}\cdot\vec{w} := v_1w_1 + v_2w_2 + \cdots + v_nw_n$$
$$= \sum_{i=1}^{n} v_i w_i$$

Basic Important Facts

- given vector $\vec{v}$ and coord vector $\vec{c}$ then $\vec{v}\cdot\vec{c} = \vec{c}\cdot\vec{v}$ = that coordinate of $\vec{v}$

  eg. $\vec{v}\cdot\vec{i} = \vec{i}$-coord, aka x-coord of $\vec{v}$

  $\vec{v}\cdot\vec{k} = z$-coord

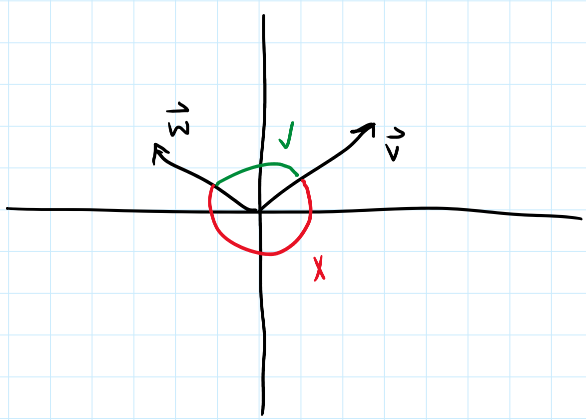
- $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$
  (aka $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$)

- bilinearity:
  say $a, b \in \mathbb{R}$
  and $\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2$ are vectors

  then
  $$(a\vec{v}_1 + b\vec{v}_2)\cdot(\vec{w}_1)$$
  $$= a(\vec{v}_1\cdot\vec{w}_1) + b(\vec{v}_2\cdot\vec{w}_1)$$
  and
  $$\vec{v}_1\cdot(a\vec{w}_1 + b\vec{w}_2)$$
  $$= a(\vec{v}_1\cdot\vec{w}_1) + b(\vec{v}_1\cdot w_2)$$

Angles

given $\vec{v}, \vec{w}$ "angle" is the smallest angle btwn them



Fact

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$$

θ = angle btwn them

$$\Rightarrow \cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$$

Note: $0 \leq \theta \leq \pi$
$0^\circ \leq \theta \leq 180^\circ$

In particular, $\vec{v}\cdot\vec{w}=0$ iff $\vec{v},\vec{w}$ are perpendicular

- Notice that if $\vec{u}$ is any vector, then $\vec{u}\cdot\vec{u} \geq 0$
  "Trivial Inequality"
  eg $\vec{u} = \vec{v} - \vec{w}$

So

$$\vec{u}\cdot\vec{u} = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$$
$$= \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w} - 2\cdot\vec{v}\cdot\vec{w} \geq 0$$

bilinearity

$$\Rightarrow \vec{v}\cdot\vec{w} \leq \frac{\vec{v}\cdot\vec{v}+\vec{w}\cdot\vec{w}}{2}$$

Note expanding $\|\vec{v}+\vec{w}\|^2$ using bilinearity gives law of cosines

Cauchy-Schwarz Inequality

$$|\vec{v}\cdot\vec{w}| \leq \sqrt{\|\vec{v}\|^2\cdot\|\vec{w}\|^2} = \|\vec{v}\|\cdot\|\vec{w}\|$$

Square both sides:

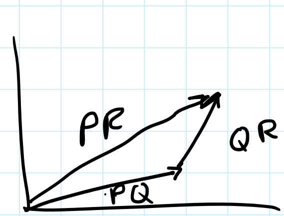
$$(\vec{v}\cdot\vec{w})(\vec{v}\cdot\vec{w}) = (\vec{v}\cdot\vec{w})^2 \leq (\vec{v}\cdot\vec{v})(\vec{w}\cdot\vec{w})$$

equivalent to: $|\cos\theta| \leq 1$

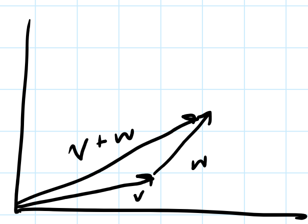
Triangle Inequality

Three equivalent forms

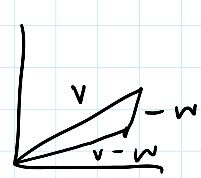
① $\|\overrightarrow{PR}\| \leq \|\overrightarrow{PQ}\| + \|\overrightarrow{QR}\|$



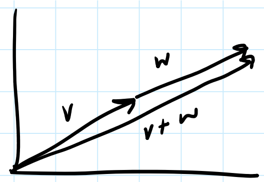
② $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$



③ $\|\vec{v}\| - \|\vec{w}\| \leq \|\vec{v}-\vec{w}\|$



Note equality in ② and ③ iff $\vec{v}$ and $\vec{w}$ in same direction



Note ② is equivalent to

$$(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \|\vec{v}+\vec{w}\|^2 \leq (\|\vec{v}\|+\|\vec{w}\|)^2$$

use bilinearity on the left, this is just

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\vec{v}\cdot\vec{w} \leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|$$

equivalent to Cauchy-Schwarz

Cross Product

$\vec{v} = (v_1, v_2, v_3)$ $\quad\quad$ $\vec{w} = (w_1, w_2, w_3)$

$$\vec{v}\times\vec{w} = (v_2w_3 - v_3w_2, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1)$$

$$\begin{array}{cccc} \vec{v} & v_1 & v_2 & v_3 \\ \vec{w} & w_1 & w_2 & w_3 \end{array}$$

$$\rightarrow \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}_{①}, \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}_{②}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}_{③} \right)$$

$$\vec{v}\times\vec{w} : \left(\underbrace{\begin{vmatrix} v_2 & v_3 \\ w_2 & w_2 \end{vmatrix}}_{\text{Determinant}}, \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$$

same as:

$$-\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}$$

"Anti-Symmetric"

$$\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$$

Determinants

Say $\vec{u} = (u_1, u_2, u_3)$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

cofactor expansion (expansion by minors)

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= (u_1, u_2, u_3) \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$$

$$= \vec{u} \cdot (\vec{v} \times \vec{w})$$

↳ combining dot product & cross product gives determinant

$= \pm$ volume of the parallelpiped determined by $\vec{u}, \vec{v}, \vec{w}$

<u>Note</u>: Determinant is always $\emptyset$ if 2 columns are the same

so $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$

$\Rightarrow$ $\vec{v}$ is perpendicular to $\vec{v} \times \vec{w}$

& $\vec{w} \perp$ to $\vec{v} \times \vec{w}$

<u>Direction</u> of cross product is that it is perpendicular to $\vec{v}$ and $\vec{w}$
(ie perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$)

perpendicular to $\vec{v}$ and $\vec{w}$
(ie perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$)

What if $\vec{v}$ & $\vec{w}$ don't span a plane, ie, they are parallel?
Then $\vec{v} \times \vec{w} = \emptyset$

Fact $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$

θ = angle btwn them

Remark $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\|$ iff $\vec{v}$ and $\vec{w}$ are $\perp$

Notice θ = angle btwn $\vec{v}, \vec{w}$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

$$\cos^2\theta = \frac{(\vec{v} \cdot \vec{w})^2}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}$$

$$\sin\theta = \frac{\|\vec{v} \times \vec{w}\|}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

$$\sin^2\theta = \frac{(\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w})}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}$$

$$\Rightarrow \frac{(\vec{v} \cdot \vec{w})^2 + (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w})}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})} = 1$$

$$\Leftrightarrow \underbrace{(\vec{v} \cdot \vec{w})^2 + (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w}) = (\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}_{\text{gives error term of Cauchy Swarz}}$$

because C-S says:
$(\vec{v} \cdot \vec{w})^2 \leq (\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})$
and now we know that
the difference/error = $(\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w})$

Note $\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram determined by $\vec{v}$ and $\vec{w}$

Half of it is the area of the triangle

Half of it is the area of the triangle

$\vec{v} \times \vec{w}$ is <u>bilinear</u> in $\vec{v}$ and $\vec{w}$

ie. given $a, b \in \mathbb{R}$

$\vec{v}_1, \vec{w}_1, \vec{v}_2, \vec{w}_2$ vectors

$(a\vec{v}_1 + b\vec{v}_2) \times \vec{w}_1$

$= a(\vec{v}_1 \times \vec{w}_1) + b(\vec{v}_2 \times \vec{w}_1)$

and similarly ....

symmetric

⚠ Recall $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

anti-symmetric

Consider

$\vec{r} = \vec{u} \times (\vec{v} \times \vec{w})$

<u>Recall</u> $\vec{v} \times \vec{w} \perp \vec{v}$

$\vec{v} \times \vec{w} \perp \vec{w}$

$\vec{r} \perp \vec{u}$

$\vec{r} \perp \vec{v} \times \vec{w}$

As long as $\vec{v} \times \vec{w} \neq 0$, $\vec{v} \times \vec{w} \perp$ to plane spanned by $\vec{v}$ and $\vec{w}$

Also, the plane spanned by $\vec{v}, \vec{w}$ is set of vectors that are $\perp$ to $\vec{v} \times \vec{w}$

$\Rightarrow \vec{r}$ is in the plane spanned by $\vec{v}$ and $\vec{w}$

$\Rightarrow \vec{r}$ is $a\vec{v} + b\vec{w}$ for $a, b \in \mathbb{R}$

$a = \vec{u} \cdot \vec{w}$

$b = -\vec{u} \cdot \vec{v}$

"linear geometry"

Part 1 General Facts abt geometry ?
dimension

- line : 1-D
- plane, $\mathbb{R}^2$: 2-D
- space, $\mathbb{R}^3$: 3-D
- point : 0-D

Q: What does it mean to be d-dim?
A: To be parametrized by d independen parameters

not req. to satisfy any eq

e.g. line paramet. by 1 indep. (usually called t) is parametrized by 2 parameters (usually called t, s or t_1, t_2)

line might be given by
$(x,y,z) = (3+t, 1+2t, 1-4t)$
↳ vector lurking $(3,1,1)$ and $(1,2,-4)$

e.g. plane might be given by
$(x,y,z) = (3+t-s, 1-t+3s, 4-3s)$

Vectors: $\underbrace{(3,1,4)}_{\text{const.}}$ and $\underbrace{(1,-1,0)}_{t}$ and $\underbrace{(1,3,-3)}_{s}$

Principle: more parameters = higher dim
e.g. $\varnothing$-parameters, e.g. $(x,y,z) = (2,7,-4)$
↖ a point!

e.g. 3-parameters
$(x,y,z) = (2+t_1-t_2, 3+t_3, 1-4t_1)$

this dermines all of space

Can also define geometric object as solution set to an eq.

Principle: more eq. = smaller dim

- Line given by 2 eq.
- Plane given by 1 eq.

e.g. $3x - 2y + z = 7$ defines a plane

eg $(2, 0, 1)$ is a point on it

eg. $\left.\begin{aligned} 3x - 2y + z &= 7 \\ x - 5y - 3z &= 4 \end{aligned}\right\}$

common soln set to these eq is a line
$\Rightarrow$ this line lies on that plane

Notice these are linear eqns

Consider non-linear eqns:

$$x^2 - 3y^2 - 4z^2 = 2$$

or $$y^2 - x^3 - x = 7$$

these give surfaces, so 2-dim, but not a plane

These eqns are algebraic
$\Rightarrow$ algebraic geometry

Also use differentiable fcns like

$$\sin^2 x - e^z + y = 2$$

$\Rightarrow$ differential geometry

Caveat usually 2 linear eqs determine a line but, consider the pair of eqs

$$\left.\begin{aligned} 3x - 2y + z &= 7 \\ 4y - 2z - 6x &= -14 \end{aligned}\right\}$$

$\rightarrow$ This determines a plane, not a line

More precise principle
more eqns that are independent from each other means lower dimension.

eg 3 independent eqs, in 3 vars determines a point eg 3 independent eqs, in 3 vars determines a point

Part 2 Lines & Parametric Eqs

Recall $(x,y,z) = (3+t, 1+2t, 1-4t)$

vectors $(3,1,1)$ and $(1,2,-4)$

$\vec{r} = (3,1,1)$ ← point on the line

$\vec{v} = (1,2,-4)$ ← direction of the line

Important If you scale $\vec{v}$, that doesn't change the line, it just changes the parametrization

How to write as a soln to 2 eqns?
In each coord of:
$(x,y,z) = (3+t, 1+2t, 1-4t)$
There is a t!

→ we can solve:
$t = x-3$
$t = \frac{y-1}{2}$
$t = \frac{z-1}{-4} = \frac{1-z}{4}$

But all the same t.

⇒ If (x,y,z) is on the line, then:

$$x-3 = \frac{y-1}{2} = \frac{1-z}{4}$$

and conversely.

"symmetric form"

Caveat: If one of the coords of $\vec{v}$ is $\emptyset$, the symmetric form looks a little different

$\vec{r} = (3,1,1)$ $\quad z = 1 + 0t = 1$
$\vec{v} = (1,2,0)$

$$\boxed{x-3 = \frac{y-1}{2} \quad \text{AND} \quad z = 1}$$ ← symmetric form

What about writing a line through

What about writing a line through points P & Q?

Then set $\vec{v} = \overrightarrow{PQ}$

and $\vec{r} = P$ (as a vector)

or $\vec{r} = Q$ (as a vector)

eg. $P = (1, 1, 2)$ $\quad Q = (2, 0, 3)$

→ Then $\vec{v} = (1, -1, 1)$

$\left.\begin{array}{l} \vec{r} = (1, 1, 2) \\ \text{or} \\ \vec{r} = (2, 0, 3) \end{array}\right\} \to$ get the same line

Similarly $\vec{v} = (-1, 1, -1)$ also gives the same line

(see Ex 1.19 in text)

Note: If L_1 is given by:

$$(x, y, z) = \vec{r}_1 + t_1 \vec{v}_1$$

and L_2 by:

$$(x, y, z) = \vec{r}_2 + t_2 \vec{v}_2$$

then $L_1 \parallel L_2$ iff $v_1 \parallel v_2$

and $L_1 \perp L_2$ iff $v_1 \perp v_2$

Note: In 2-space (Euclidean plane) the two lines either

1. intersect
2. are parallel

But in $\mathbb{R}^3$, they can also be skew ie, not parallel & don't intersect

← example

Note, if $L_1 \perp L_2$ but don't intersect they are skew

([Co] 1.22)

Distance from a point to a line

Distance from a point to a line

Given point P and line L given by $\vec{r} + t\vec{v}$.

Then: The distance is

$$\frac{\|\vec{w} \times \vec{v}\|}{\|\vec{v}\|}$$

where $\vec{w}$ is the vector from $\vec{r}$ to P

eg. if $P = (2,3,1)$ & $\vec{r} = (3,0,-1)$
$\Rightarrow \vec{w} = (-1,3,2)$

Some symmetry

- If we scale $\vec{v}$ (replace $\vec{v}$ by $2\vec{v}$) then $\frac{\|\vec{w} \times \vec{v}\|}{\|\vec{v}\|}$ doesn't change
- If we replace $\vec{w}$ by $-\vec{w}$ (eg take $\vec{w}$ to be from P to $\vec{r}$), then:

$$\frac{\|\vec{w} \times \vec{v}\|}{\|\vec{v}\|}$$

- If we choose a diff point $\vec{r}$ on the same line, the formula gives the same answer

Why? eg

replace $\vec{r}$ by $\vec{r} + 3\vec{v}$ then the effect $\vec{w}$ is to replace $\vec{w} = P - \vec{r}$

by $P - (\vec{r} + 3\vec{v})$

$= \vec{w} - 3\vec{v}$

then, if we replace $\vec{w}$ by $\vec{w} - 3\vec{v}$ in

$$\frac{\|\vec{w} \times \vec{v}\|}{\|\vec{v}\|},$$ we get

$$\frac{\|(\vec{w} - 3\vec{v}) \times \vec{v}\|}{\|\vec{v}\|}$$

$$\frac{\|(\vec{w} - 3\vec{v}) \times \vec{v}\|}{\|\vec{v}\|}$$

but
$$(\vec{w} - 3\vec{v}) \times \vec{v} = \vec{w} \times \vec{v} - 3\vec{v} \times \vec{v}$$
by bilinearity
$$= \vec{w} \times \vec{v} - \emptyset$$

(think 1.4 problem (27b))

Part 3: Planes

Recall: plane defined by 1 eqn

eg.
$ax + by + cz = d$ "normal form"

Notice can rewrite using dot prod.

$\vec{r} \cdot (a, b, c) = d$

with $\vec{r} = (x, y, z)$

Better way

1) Choose point on the plane $(x_0, y_0, z_0) = \vec{r}_0$

2) $\vec{r}_0 \cdot (a, b, c) = d$

$\Rightarrow$ we can rewrite eqn as

$\vec{r} \cdot (a, b, c) = \vec{r}_0 \cdot (a, b, c)$

equivalently:

$\vec{r} \cdot (a, b, c) - \vec{r}_0 \cdot (a, b, c) = 0$

3) Use bilinearity

$(\vec{r} - \vec{r}_0) \cdot (a, b, c) = 0$

ie, this eq. just says that $\vec{r} - \vec{r}_0 \perp (a, b, c)$

So $\vec{r}$ is in this plane iff $\vec{r} - r_0 \perp (a, b, c)$
"point-normal form"

How to get plane cont 3 points P, Q, R?

→ Can write parametrically as

$$(x, y, z) = \vec{P} + t\,\overrightarrow{PQ} + s\,\overrightarrow{PR}$$

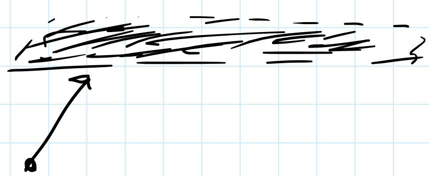
Q: What if we want a normal vector?
ie — vector $\perp$ to $\overrightarrow{PQ}$ & $\overrightarrow{PR}$?

A: Cross-product
(Ex 1.2.4)

See formula 1.27 — distance from a point to a plane

→ can think of as $\dfrac{\|\vec{w} \cdot \vec{n}\|}{\|\vec{n}\|}$

for $\vec{n}$ normal vector and $\vec{w}$ from any point on plane to chosen point P.



— Read book for line of intersection of two planes.

Sect 1.2

1e) $\vec{v} = (-1, 5, -2)$

$\vec{w} = (3, 1, 1)$

Find $\left\| \frac{1}{2} (\vec{v} + \vec{w}) \right\|$

$\vec{v} + \vec{w} = (2, 6, -1)$

$$\vec{v} + \vec{w} = (2, 6, -1)$$

$$||\vec{v} + \vec{w}|| = \sqrt{2^2 + 6^2 + (-1)^2}$$

$$= \sqrt{4 + 36 + 1}$$

$$= \sqrt{41}$$

$$\frac{1}{2}||\vec{v} + \vec{w}|| = \frac{\sqrt{41}}{2}$$

Surfaces

plane — linear surface

Spheres

Sphere Equation

$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$] Cartesian

Vector eq:

$\vec{x}=(x,y,z)$ $\quad \vec{x_0}=(x_0,y_0,z_0)$

$r^2=\|\vec{x}-\vec{x}_0\|^2=(\vec{x}-\vec{x}_0)\cdot(\vec{x}-\vec{x}_0)$

= sq. of dist from $\vec{x}$ to $\vec{x_0}$

$=\vec{x}\cdot\vec{x}+\vec{x}_0\cdot\vec{x}_0-2\vec{x}\cdot\vec{x}_0$

$-x^2+y^2+z^2+ax+by+cz+d=\emptyset$

Q Given such an eq, does it define a sphere?

A Not necessarily

e.g. $x^2+y^2+z^2+1=\emptyset$

$\iff x^2+y^2+z^2=-1$

$\Rightarrow$ empty set in $\mathbb{R}^3$

e.g. $-x^2+y^2+z^2=\emptyset$

$\Rightarrow$ point (ie sphere of radius $\emptyset$)

- Given $x^2+y^2+z^2+ax+by+cz+d$, complete the square to figure out what it is

Intersections

sphere and a line: get 0, 1, or 2 pts

Algebraically, easiest way to solve is to write in parametric form

$\vec{x}=(x,y,z)=\vec{x_1}+t\vec{v}$

then plug this into the eq for sphere to get a quadratic eq from it

$\vec{x}_1$ = pt on line, $\vec{x_0}$ = center of sphere

→ Eq for t

$\vec{x}=\vec{x}_1+t\vec{v}$

→ Eq for t

$\vec{x} = \vec{x}_1 + t\vec{v}$

$r^2 = (\vec{x} - \vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$

$= (t\vec{v} + \vec{x}_1 - \vec{x}_0) \cdot (t\vec{v} + \vec{x}_1 - \vec{x}_0)$

$= (\vec{v} \cdot \vec{v})t^2 + 2(2(\vec{x}_1 - x_0) \cdot \vec{v})t + (\vec{x}_1 - \vec{x}_0) \cdot (\vec{x}_1 - \vec{x}_0)$

Sphere and a Sphere

Intersection is either:

1. a circle
2. a point
3. empty

Easier way to find intersection

$x^2 + y^2 + z^2 + a_1x + b_1y + c_1z + d_1 = \emptyset$

$x^2 + y^2 + z^2 + a_2x + b_2y + c_2z + d_2 = \emptyset$

→ subtract top from bottom

$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z + d_1 - d_2 = 0$

⇒ eq of a plane

⇒ intersection of the spheres is the intersection of one sphere with that plane

→ solve for one of x, y, or z then plug into the other eq

eg. if $a_1 - a_2 \neq 0$, can solve for x

but if $a_1 - a_2 = 0$, solve for y

Q what if $a_1 - a_2 = b_1 - b_2 = c_1 - c_2 = \emptyset$?

A this happens if the two spheres are concentric

then, either:

1. radii are different

→ empty intersection

2. radii are same

→ they are the same sphere

→ intersection in a sphere

Cylinder

e.g. $(x-a)^2 + (y-b)^2 = r^2$

eg $(y-b)^2 + (z-c)^2 = r^2$ another cylinder

Intersections

Cylinder ∩ xy plane

its intersection w/ xy-plane is a

its intersection w/ xy-plane is a circle of radius r.

↳ Def

the intersection of a surface w/a plane is a trace of that surface

---

Quadric Surfaces

↳ Def

Anything given by an eqn of the form

$ax^2 + by^2 + cz + dxy + eyz + fxz + gx + hy + iz + j = \emptyset$

↳ Examples

- sphere & cylinder
- ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

equivalently:

$$\alpha x^2 + \beta y^2 + \gamma z^2 = \delta$$

$\delta \neq \emptyset$

can be put into the form above

Traces are ellipses

- hyperboloid

① one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

② two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces of hyperboloids are conic sections

ie — ellipses, hyperbola & parabola

Q How to find?

On a plane parallel to coord plane just set x, y, or z to be a constant and then get eq of trace in the other var just set x, y, or z to be a constant and then get eq of trace in the other var.

<u>eg</u> One-sheet hyperboloid

→ if we set $z = $ const

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$$

⇒ ellipse

→ if we set $y = $ const

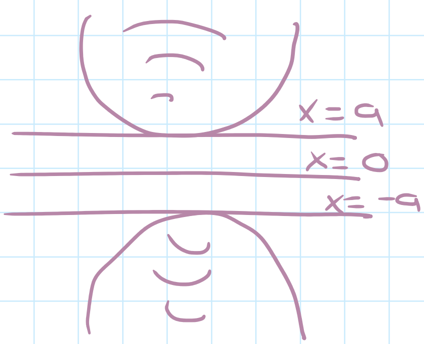
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$

<u>eg</u> two-sheet

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2} - 1$$

→ if set $x = $ const
(ie, plane ∥ to yz plane)
then either



btwn the sheets → ① $x^2 - a^2 < 0$
→ trace is empty

tangent to one sheet ② $x^2 - a^2 = 0$
→ trace is a point

③ $x^2 - a^2 > 0$
→ trace is an ellipse

see also: elliptic paraboloid.
hyperbolic paraboloid

- <u>elliptic cone</u>:
like 2 cones, one of them upside down meeting at a point

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

<u>Remark</u>

x^2 , y^2 z^2 = d

Remark

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=d$$

defines
1. hyperboloid of one sheet if $d>0$
2. hyperboloid of two sheets if $d<0$
3. elliptic cone if $d=0$

Ruled Surfaces

↳ def

A surface is ruled for any point P on the surface, there's

e.g. a cylinder is ruled

Given any (x_0, y_0, z_0) on the cylinder $(x-a)^2+(y-b)^2=r^2$ the line given by the two eqns $x=x_0$, $y=y_0$ and is contained in the cylinder

BUT sphere is not ruled.

Q. Why?

A. bc no line is contained wholly in the sphere

Doubly ruled

↳ given any point, there are two distinct lines through that point contained in the surface

"regulus"

Curvilinear Coordinates

Cylindrical coords

Cylindrical Coords

↳ defined by

(r, θ, z) such that $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$ $\quad r^2 = x^2 + y^2$

$\theta = \arcsin(y/r)$

→ like polar coord in x, y & don't do anything to z

Note $r \geq \emptyset$
$\emptyset \leq \theta \leq 2\pi$

Q: Why "cylindrical"?
A: b/c an eqn $r = \text{const}$ defines a cylinder

cool geometric surface:
$z = \theta$ defines a "helicoid"
→ looks like parking garage

Spherical Coords (ρ, θ, ϕ) → (rho, theta, phi)

θ ← same as cylindrical

$\rho^2 = x^2 + y^2 + z^2$

$\rho = \|(x, y, z)\|$

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \cos\theta$
$z = \rho \cos\phi$

ϕ = angle from z-axis

↳ so $\phi = \emptyset$ if on positive z-axis
$\phi = \pi$ if on negative z-axis
$\phi = \pi/2$ if on xy-plane

$$\boxed{\begin{aligned} 0 \leq \theta < 2\pi \\ 0 \leq \phi \leq \pi \end{aligned}}$$

↳ so $\boxed{r = \rho \sin\phi}$ relation btwn spherical & cylindrical coords

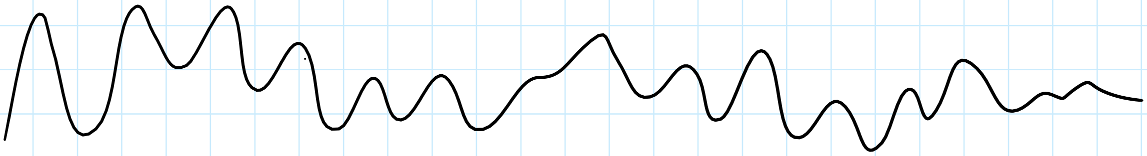
Q: Why "spherical"?
A: Bc eq $\rho = \text{const}$ defines a sphere centered at origin.

centered at origin.

Q: What about sphere centered somewhere else?
A: See Example 1.33
(The eqn is REALLY messy)

Helix

$z = \theta \quad r = \text{const}$
$\Rightarrow$ a curve



Problems

Sec 1.2 C6

Q Can every vector in $\mathbb{R}^3$ be written as a linear combo of $\vec{i}$ and $\vec{j}$?
i.e. $\vec{v} = m\vec{i} + n\vec{j}$?

A no
eg $(0,0,1) = \vec{k}$ cannot be written this way

Why?
If $\vec{v} = m\vec{i} + n\vec{j}$ then the z coord of $\vec{v}$ must be $\emptyset$
Therefore if $\vec{v}$ has nonzero z-coord, then $\vec{v}$ cannot be written in that form

Sec 1.3 A6

Q angle btw $\underbrace{(4,2,-1)}_{=\vec{v}}$ & $\underbrace{(8,4,-2)}_{=\vec{w}}$

notice $\vec{w} = 2\vec{v}$
$\hookrightarrow$ $\vec{w}$ & $\vec{v}$ point in same dir
$\rightarrow$ angle btw them $= \emptyset$

1.8 Vector-Valued Functions

Tuesday, February 2, 2021 12:17 PM

Naive definition: Vector-valued fcn is a fcn from $\mathbb{R}$ to $\mathbb{R}^3$

i.e. $\vec{f}$ sends $t \in \mathbb{R}$ to $\vec{f}(t) \in \mathbb{R}^3$

But what about:

$\vec{f}(t) = (t, 3t, 1/t)$

$\rightarrow$ not defined at $t = 0$

so it's a fcn from $\mathbb{R} \setminus \{0\}$ = set of nonzero real numbers to $\mathbb{R}^3$

Better definition: A vector-valued fcn is a fcn from a subset D of $\mathbb{R}$ to $\mathbb{R}^3$.

e.g. $\vec{f}(t) = \left(\frac{1}{1-t}, \sqrt{t}, \sin(t)\right)$

is defined for $t \geq 0$ and $t \neq 1$.

i.e. $D = [0,1) \cup (1, \infty)$

= set of real numbers that are neither negative nor equal to 1.

Can think of as a parametric eq. in $\mathbb{R}^3$

e.g.

line: $\vec{f}(t) = \vec{x_0} + t\vec{v}$

helix: $\vec{f}(t) = (\cos t, \sin t, t)$

Can write vector-val'd fcn as:

1. $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$
2. $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$

Note: A lot of vector valued calc is just a matter of doing single var calc in each coord separately (true of limits, continuity, derivative)

$\rightarrow$ Becomes something new when we do dot & cross products

Limits

Definition: If $\vec{F}$ is a v-v f on D and $a \in D$ and $\vec{c} \in \mathbb{R}^3$, we say:

$$\lim_{t \to a} \vec{F}(t) = \vec{c}$$

if one of 2 eq. conditions holds:

(A) $\forall \varepsilon > 0, \exists \delta > 0$
s.t. if $|t-a| < \delta$
then distance $(\vec{F}(t), \vec{c}) < \varepsilon$

$$\text{distance}(\vec{F}(t), \vec{c}) = \|\vec{c} - \vec{F}(t)\|$$

(B) for $i = 1, 2, 3$ we have
$$\lim_{t \to a} f_i(t) = c_i$$

Why are (A) & (B) equivalent?

- The ith coord of $\vec{c} - \vec{F}(t)$ is $c_i - f_i(t)$.
- Def (A) says that we can make $\|\vec{c} - \vec{F}(t)\|$ small when t is close to a.
  Def (B) says that we can make $|c_i - f_i(t)|$ small when t is close to a.
- They are equivalent bc a vector is small in magnitude iff its components are all small in absolute value.
- Qualitatively:
  For a vector $\vec{v} = (v_1, v_2, v_3)$ each of $|v_1|$, $|v_2|$, and $|v_3|$ is $\leq \|\vec{v}\|$
  and
  $\|\vec{v}\| \leq |v_1| + |v_2| + |v_3|$

Continuity

Suppose $\vec{f}(t)$ is defined as a v-v f for $t, a \in D$

Then we say $\vec{f}$ is continuous if either of the two eq. conds hold:

(A) $\lim_{t \to a} \vec{f}(t) = \vec{f}(a)$

(B) each of $f_1(t), f_2(t), f_3(t)$ is cont at a.

Derivatives

We define (for $a \in D$):

$$\vec{f}'(a) = \lim_{h \to 0} \frac{\vec{f}(a+h) - \vec{f}(a)}{h}$$

$$= \lim_{t \to a} \frac{\vec{f}(t) - \vec{f}(a)}{t-a}$$

We say $\vec{f}$ is differentiable at a if this limit exists.

equivalently

$\vec{f}$ is differentiable iff $f_1, f_2,$ and f_3 are differentiable

P iff Q means if P then Q & if Q then P.

If $\vec{f}$ is differentiable at a, then

$\vec{f}'(a) = (f_1'(a), f_2'(a), f_3'(a))$

New idea: Derivative is a vector not a scalar.

ie, has magnitude & direction

Physical Interpretation

For an obj whose position at time t is given by $\vec{f}(t)$, its velocity is $\vec{f}'(t)$, its speed is $\|\vec{f}(t)\|$, the direction of $\vec{f}'(t)$ is the direction the obj is moving.

acceleration is:

$$\vec{f}''(t) = \frac{d}{dt} \vec{f}'(t) \leftarrow \text{Check this}$$

$$\vec{F}''(t) = \frac{dF}{dt} f'(t)$$ ← Check this

In physics

Basic Properties of Deriv's

Same as in single var

① $\vec{F}(t) = \emptyset$ iff $\vec{F}$ is a constant fcn (on each interval)

- In general, for any D, if $\vec{F}$ is const, then $\vec{F}'(t) = 0$
- If D is an interval like (a,b) or $[a,b]$ or half-open, then if $\vec{F}'(t) = 0$ then $\vec{F}(t)$ is constant.

② Linearity

If $m, n \in \mathbb{R}$ and $\vec{f}$ and $\vec{g}$ are diff'able v-v f, then

$$\frac{d}{dt}(m\vec{f}(t) + n\vec{g}(t)) = m f'(t) + n g'(t)$$

. derivative of a linear combination is a linear combination of the derivatives

Different in MVC

① New kinds of products:

- multiply vector by scalar
  output: vector
- dot product of 2 vectors
  output: scalar
- cross product of 2 vectors
  output: vector

⇒ 3 product rules for derivatives

Let $\vec{f}, \vec{g}$ be v-v f on $D \subseteq \mathbb{R}$

and $u(t)$ be a scalar-valued func on D. Then:

vector

scalar

① $\frac{d}{dt}(u(t)\vec{F}(t))$

$= u'(t)F(t) + u(t)\vec{F}'(t)$

② $\frac{d}{dt}(\vec{F}(t)\cdot\vec{g}(t))$

$= \vec{F}'(t)\cdot\vec{g}(t) + \vec{F}(t)\cdot\vec{g}'(t)$

ie we wrote a single-var calc deriv in terms of dot prods of vector derivatives

③ $\frac{d}{dt}(\vec{F}(t)\times\vec{g}(t))$

$= \vec{F}'(t)\times\vec{g}(t) + \vec{F}(t)\times\vec{g}'(t)$

⚠ order of cross product matters

Let's use dot product for some vector calculus geometry

Consider speed $\|\vec{F}'(t)\|$

Actually, let's consider speed^2

$= \|\vec{F}'(t)\|^2 = \vec{F}'(t)\cdot F'(t)$

Two ways:

Ⓐ Use ①

$\frac{d}{dt}(\text{speed}^2) = \frac{d}{dt}(\vec{F}'\cdot\vec{F}')$

$= \left(\frac{d}{dt}\vec{F}'\right)\cdot\vec{F}' + \vec{F}\cdot\frac{d}{dt}(\vec{F}')$

$= \vec{F}''\cdot\vec{F}' + \vec{F}'\cdot\vec{F}''$

$= 2F'\cdot\vec{F}''$

Ⓑ $\frac{d}{dt}(\text{speed}^2)$

single-var product rule)

(B) $\frac{d}{dt}(\text{speed}^2)$ single-var product rule

$$= 2(\text{speed}) \cdot \frac{d}{dt}(\text{speed})$$

$$= 2\|\vec{F}'\| \cdot \frac{d\|\vec{F}\|}{dt}$$

$$= \frac{d}{dt}\|\vec{F}\|^2$$

Conclusion

(1) When is speed constant?

Note: speed is const iff speed² is const.

and this is the iff

$$\frac{d}{dt}\text{speed}^2 = \varnothing = 2\vec{F}' \cdot \vec{F}''$$

Q: when is $\vec{F}' \cdot \vec{F}'' = 0$?

A: When $\vec{F}' \perp \vec{F}''$

so speed doesn't change (ie, only the direction changes) precisely when (iff) the acceleration is perpendicular to the direction of motion

(2) Formula for $\frac{d}{dt}(\text{speed})$. How?

Note: set (A) equal to (B)

$$2(\text{speed})\frac{d}{dt}(\text{speed}) = \frac{d}{dt}(\text{speed}^2)$$

$$= 2\vec{F}' \cdot \vec{F}''$$

$$\Rightarrow (\text{speed}) \cdot \frac{d}{dt}(\text{speed}) = 2\vec{F}' \cdot \vec{F}''$$

$$\Rightarrow \frac{d(\text{speed})}{dt} = \frac{d\|\vec{F}'(t)\|}{dt} = \frac{2\vec{F}' \cdot \vec{F}''}{\|F'\|}$$

See in book similar reasoning with $\vec{F}$ in place of $\vec{F}'$ shows that

① $\|\vec{F}\|$ is const, ie $\vec{F}(t)$ is contained in a circle iff $\vec{F} \perp \vec{F}'$.

② $\dfrac{d\|\vec{F}(t)\|}{dt} = \dfrac{d\rho}{dt} = \dfrac{\vec{F} \cdot \vec{F}'}{\|\vec{F}\|}$

Arclength & curvature

ie. intrinsic properties of curves

e.g. circle of radius 1 given by

$(x, y, z) = (\cos t, \sin t, 0)$

or $(x, y, z) = (\cos t^2, \sin t^2, 0)$

these are 2 parametrizations of the same curve.

Q What abt $(x, y, z) = (\cos t^2, \sin t, 0)$?

A No, it's a completely different curve

Another eg

parabola in xy plane $(x, y, z) = (t, t^2, 0)$

how about $(x, y, z) = (t^3, t^6, 0)$

→ also same parabola

What about $(x, y, z) = (t^2, t^6, 0)$?

→ not just a diff parametrization of the same curve

→ this looks like $y = x^3$

In general say we have

$\vec{F}(t) = (x(t), y(t), z(t))$

and let g be a single-var function

such that: For $t \in D'$ (possibly some other domain)

① $g(t) \in D$

② g is strictly monotone increasing, i.e.

for $t_1, t_2 \in D'$ if $t_1 < t_2$

then $g(t_1) < g(t_2)$

eg $g(t) = t^2$ is monotone increasing

on $D' = [0, \infty)$

then $\vec{F}(g(t))$ is vector-valued func defined for $t \in D'$ (b/c then $g(t) \in D$ so we can give it as input to $\vec{F}$.

now $\vec{F}(t)$ and $\vec{F}(g(t))$ are parameterizations of the same curve

Recall given $\vec{F}(t)$,

velocity $\vec{F}'(t)$

acceleration $\vec{F}''(t)$

speed $\|\vec{F}'(t)\|$

suppose $\vec{F}(t)$ is defined on $[a, b]$

(i.e $[a, b] \subseteq D$)

Define a func s as follows:

for $t \in [a, b]$, let $s(t)$ denote

Define a func s as follows:
for $t \in [a,b]$, let $s(t)$ denote the distance the obj has traveled since $t=a$

So $S(t)$ is a nonnegative real #.

Q What is $s(a)$?

A As t goes from a to b, $s(t)$ generally increases as long as the obj isn't stationary, ie as long as $\vec{F}'(t) \neq 0$.
[In that case, s is monotonically increasing]
(More generally, if $\vec{F}'(t) = 0$ at a single point but is nonzero everywhere else, then monot incr.)

Qualititatively, we know
① $s(a) = 0$
② s increases (or stays same if $\vec{F}(t) = 0$) as t goes from a to b.

Q Quantitatively, how to compare s?

A deriv ds/dt
is how much speed change/time
$$\frac{ds}{dt} = ||\vec{F}(t)||$$

So we know:
① $s(a) = 0$
② $\frac{ds}{dt} = ||\vec{F}'(t)||$

Using FTC,
$$s(t_1) = \int_a^{t_1} ||\vec{F}'(t)|| dt$$

To compute $s(t)$, you have to
① compute a derivative
② Find a magnitude (as fcn of t)
③ compute a single-var integral

e.g. $\vec{F}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$
(circle of radius 1)
① $\vec{F}'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j}$
② $||F'(t)|| = \sqrt{(-\sin t)^2 + \cos^2 t}$
$= \sqrt{\sin^2 t \cos^2 t} = 1$

③ Length from a to b

③ Length from a to b

$$s(b) = \int_a^b 1 dt = b-a$$

=> length of sector of circle of radius 1 is the angle (in radians) of the sector.

* see book for helix *

Another way to write arc length

$$s(t_1) = \int_a^{t_1} ||\vec{F}'(t)|| dt$$

why "t_1"?

(Compare in single var:

$$F(t_1) = \int_a^{t_1} f(t) dt$$)

$$= \int_a^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

$$= \int_a^{t_1} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right](dt^2)}$$

$$= \int_a^{t_1} \sqrt{dx^2 + dy^2 + dz^2}$$

"intrinsic form"

→ independent of parameterization

Can prove that s is independent of parameterization using chain rule

Suppose $\vec{F}(t)$ defined for

$t \in [a,b] \subseteq D$.

Say $[c,d] \subseteq D'$ and g maps D' to D.

(and is strictly monot. incr)

suppose $g(c) = a$
and $g(d) = b$

!!!

"the interval $[c,d]$ corresponds under change of parameterization to $[a,b]$"

1st parameterization	$\vec{F}(t)$ $t \in D$
2nd parameterization	$\vec{F}(g(t))$ $t \in D$

$t = a$ in 1st param corresponds to
$t = c$ in 2nd
i.e. $\vec{f}(a) = \vec{f}(g(c))$

Q What do we mean when say arc length independent of parameterization.

A We mean that

$$\int_a^b \|\vec{f}'(t)\| dt = \int_c^d \|(\vec{f} \circ g)'(t)\| dt$$

Q Why?

A Follows by chain rule (integration by substitution)

Arc Length Parameterization

Recall As long $\vec{F}(t)$ doesn't remain const for period of time, $s(t)$ is strictly monotonically increasing in t.

$\Rightarrow$ can use it for reparametrization

choose g to be inverse fcn of s $\leftarrow$ relative to some initial pt $t = a$
i.e. $g(s(t)) = t$

Now

Consider $\vec{F} \circ g$ and input arc length, then we get corresponding $\vec{f}$.

e.g. circle $(\cos t, \sin t, 0) = \vec{f}(t)$
We know arclength from 0 is t.

Arclength from $t=a$ is $t-a$.
so $s(t) = t-a$
$\Rightarrow g(t) = t+a$
(just as $s(a)=0$, also $g(0)=a$)

Try circle radius 2
$\vec{f}(t) = (2\cos t, 2\sin t, 0)$

Now
$s(t) = 2(t-a)$

What's $g(t)$?
$g(t) = t/2 + a$

(another way to write: $t = \frac{s}{2} + a$)

Now What's $\vec{F}(g(t))$?
A $(2\cos(g(t)), 2\sin(g(t)), 0)$
$= (2\cos(\frac{t}{2}+a), 2\sin(\frac{t}{2}+a), 0)$

this is the parametrization by arc length starting at a.

ie $t=a$ in og parametrization corresponds to $t=0$ in the arc length parametrization

Cylindrical coords

Recall $x = r\cos\theta$ $y = r\sin\theta$ $z = z$

And $s = \int \sqrt{dx^2 + dy^2 + dz^2}$

(another way to write:
$ds^2 = dx^2 + dy^2 + dz^2$)

Idea Apply d to both sides of $x = r\cos\theta$
$\Rightarrow dx = d(r\cos\theta) = (dr)(\cos\theta) + r(d\cos\theta)$

and $d\cos\theta = d\theta\left(\frac{d\cos\theta}{d\theta}\right) = -\sin\theta\, d\theta$

so $dx = dr(\cos\theta) + r(-\sin\theta\, d\theta)$
$= \cos\theta\, dr - r\sin\theta\, d\theta$

$dy = d(r\sin\theta) = \sin\theta\, dr + r\, d\sin\theta$

$$= \sin\theta \, dr + r\cos\theta \, d\theta$$

$\Rightarrow dx^2 + dy^2$

$$= (\cos\theta \, dr - r\sin\theta \, d\theta)^2 + (\sin\theta \, dr + r\cos\theta \, d\theta)^2$$

$$= \cos^2\theta \, dr^2 + \sin^2\theta \, dr^2 + r^2\sin^2\theta \, d\theta^2 + r^2\cos^2\theta \, d\theta$$

$$= dr^2 + r^2 d\theta^2$$

$\Rightarrow dx^2 + dy^2 + dz^2 = 0$

$$= dr^2 + r^2 d\theta^2 + dz^2$$

$\Rightarrow s = \int \sqrt{dx^2 + dy^2 + dz^2}$ intuitive/abstract

$$= \int \sqrt{dr^2 + r^2 d\theta + dz^2}$$ For computation

in terms of t $= \int \sqrt{\left(\frac{dr}{dt} dt\right)^2 + r^2\left(\frac{d\theta}{dt} dt\right)^2 + \left(\frac{dz}{dt}\right) dt^2}$

Read [CH] for next time $= \int dt \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

What is curvature?

It should be a quantity describing qualitative visual property of how much a path is curved.

Eg — 0 for a line (iff $\emptyset$ curvature)

— non-zero for circle
- small for large radius circle
- large for small radius circle

curvature = "dizziness factor"

Notice - a curve is a line iff curvature $= \emptyset$
- $y = f(x)$ is a line iff 2nd derivative $f''(x) = 0$

Q Is curvature just like 2nd deriv? (eg if $(x, y) = (t, f(t))$)

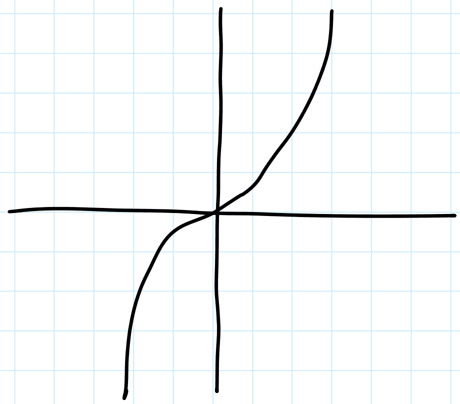
A No.

eg $y = x^3$ then for x large,

$\frac{d^2y}{dx^2}$ is large, but curvature is small

Curvature how much the path of a vector-valued function "curves"
- zero iff the path is a line
- why not 2nd derivative?

Consider $y = x^3$ (in param: $(t, t^3) = (x, y)$)



2nd deriv $6x$

So, as x gets large, so does the 2nd deriv

Curvature?
Gets small as x gets really large bc path becomes really close to being a vertical line

Idea 2nd derivative (acceleration)
- measures change in velocity
Two ways velocity can change:
1. change direction } → curvature
2. change speed

Given $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$\vec{r}\,' = \frac{d\vec{r}}{dt} = \vec{v}$ is velocity

speed: $\|\vec{v}\|$
direction: $\frac{\vec{v}}{\|\vec{v}\|} = \vec{T}$ ← unit tangent vector

try $\frac{d\vec{T}}{dt}$

Problem this depends on parameterization
try $\frac{d\vec{T}}{ds}$

Problem this is a vector

Try
$\underbrace{\left\| \frac{d\vec{T}}{ds} \right\|}_{\kappa}$ ← this is curvature!

κ = kappa

What about direction of $\frac{d\vec{T}}{ds}$?

Define $\vec{N} = \frac{d\vec{T}/ds}{\|d\vec{T}/ds\|}$

so $\frac{d\vec{T}}{ds} = \kappa\vec{N}$

See [Co] Sec. 1.9, prob 10 for κ in terms of $\vec{r}(t)$ (aka $\vec{r}(t)$)

Why "$\vec{N}$"?

Because: $\vec{T}\cdot\vec{T} = 1$ is const

if we d/ds both sides

$$2\vec{T}\cdot\frac{d\vec{T}}{ds} = \frac{d}{ds}\vec{T}\cdot\vec{T} = \frac{d}{ds}1 = 0$$

So $\vec{T}\cdot\frac{d\vec{T}}{ds} = 0$ so $\frac{d\vec{T}}{ds}$ is perpendicular to $\vec{T}$

Similarly, since $\vec{N} = \frac{d\vec{T}/ds}{\text{scalar}}$ is in the same direction as $d\vec{T}/ds$, $\vec{N}$ is also perpendicular to $\vec{T}$.

$\rightarrow$ N stands for Normal

"unit normal vector"

So $\vec{N}$ is the direction in $\vec{T}$ is changing

Now, in 3-space ($\mathbb{R}^3$), we can consider the plane spanned by $\vec{T}$ and $\vec{N}$

"the plane in which the object is infinitesimally moving in"

- so if $\vec{r}(t)$ stays in that plane, T & N are in that plane
- if $\vec{r}(t)$ doesn't stay in a plane, then the plane spanned by $\vec{T}$ and $\vec{N}$ changes over time

Torsion (τ) = measure of how much the plane is changing

How to measure?

Define $\vec{B} = \vec{T}\times\vec{N}$

Notice since $\|\vec{T}\| = \|\vec{N}\| = 1$ and $\vec{T}\cdot\vec{N} = 0$, also $\|\vec{B}\| = 1$

By considering $\frac{d}{ds}(\vec{B}\cdot\vec{B})$, we find that $\frac{d\vec{B}}{ds}$ is $\perp$ to $\vec{B}$

rough idea τ = torsion

$$= \|d\vec{B}/ds\|$$

Problem want to allow τ to be negative

Better idea

Notice $\frac{d\vec{B}}{ds}$ is $\perp$ to $\vec{B}$ and to $\vec{T}$ $\Rightarrow$ parallel to $\vec{N}$.

$$\therefore \frac{d\vec{B}}{ds} = (\text{scalar})\cdot\vec{N}$$

Define τ by

$$\frac{d\vec{B}}{ds} = -\tau \vec{N}$$

See 4.3 [CH]

eg $\tau \neq 0$ for a helix

Why is $\frac{d\vec{B}}{ds} \perp \vec{T}$?

$\vec{B} \cdot \vec{T} = 0$ d/ds both sides

$$\frac{d\vec{B}}{ds} \cdot T + B \cdot \frac{d\vec{T}}{ds} = 0$$

and $\frac{d\vec{T}}{ds} \parallel \vec{N}$ so $B \cdot \frac{d\vec{T}}{ds} = 0$

* Ignore Maple calculations in [CH] *

Functions of Multiple Variables [Co] 2.1

e.g.

$f(x,y) = xy$ defined $\forall (x,y) \in \mathbb{R}^2$

$f(x,y) = \dfrac{1}{x-y}$ defined for only some $(x,y) \in \mathbb{R}^2$

Defined on some subset $D \subseteq \mathbb{R}^2$

→ in this case when $x \neq y$

So $D = \{(x,y) \in \mathbb{R}^2 \mid x \neq y\}$

set of (x,y) in $\mathbb{R}^2$ — such that

$= \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \mid x = y\}$

"points in $\mathbb{R}^2$ not on the line $x = y$"

Defn A real-valued fcn $f(x,y)$ assigns a real # to every $(x,y) \in D \subseteq \mathbb{R}^2$

↳ D is domain of f

(If n-vars, $f(x,y\ldots,n)$ and $D \subseteq \mathbb{R}^n$)

Back to 2 vars:

Geometrically its graph is a surface in $\mathbb{R}^3$ (Just like graph of $y = f(x)$ is a curve in $\mathbb{R}^2$)

The points of the graph are $(x, y, f(x,y))$ for $(x,y) \in D$.

Level curves Given $f(x,y)$ and $c \in \mathbb{R}$, the level curve is the set of $(x,y) \in D$ such that $f(x,y) = c$.

In set notation: $\{(x,y) \in D \mid f(x,y) = c\}$

Notice: if f is a const fcn
eg $f(x,y) = 4$
The level curve is:

$\emptyset$ (empty set) if $c \neq 4$
$\mathbb{R}^2$ (whole plane) if $c = 4$

egs where it is a curve

- $f(x,y) = 3x - 2y$
  Then all the level curves are lines perpendicular $(3,-2)$
  [As you change c, you get diff lines, but all are $\parallel$ to each other]
- $f(x,y) = x^2 + y^2$
  the level curve is a circle if $c > 0$
  point if $c = 0$
  $\emptyset$ if $c < 0$

For any single variable func g, set $f(x,y) = y - g(x)$
Then level curve w/ $c = 0$ is graph of g

Note: Level curves are traces of the graph of $z = f(x,y)$ on horizontal planes

---

Limits & Continuity

Limits say $f(x,y)$ defined "near (a,b)" but not necessarily at (a,b)

Formally suppose $f(x,y)$ is def'd in a "punctured neighborhood" of (a,b)
i.e. a set of the form

$$\{(x,y) \in \mathbb{R}^2 \mid \underset{\substack{\uparrow \\ \text{"punctured"}}}{0 <} \|(a,b) - (x,y)\| < \varepsilon\}$$

for some $\varepsilon > 0$

We say:

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if $\forall \varepsilon > 0 \quad \exists \delta > 0$

such that if $\qquad$ but $(x,y) \neq (a,b)$

$\|(x,y) - (a,b)\| < \delta$

then

$|f(x,y) - L| < \varepsilon$

Intuitively as (x,y) gets closer to (a,b), $f(x,y)$ gets closer to L.

Caveat must be true no matter which direction (x,y) approaches (a,b)

e.g.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \quad \text{DNE}$$

Why?

If you approach along x or y axis, then it seems the limit is 0 bc $f(x,y)$ gets closer; closer to 0

BUT if $(x,y) \to (0,0)$ along $y = x$, then $f(x,y)$ approaches $1/2$

$f(x,y) = \sin\theta\cos\theta$ for (θ, r) polar coords

Basic properties of limits are the same (addition, sub, mult, div)

↳ as long as denom does not approach 0.

Continuity

Suppose $f(x,y)$ is defined for (x,y) near (a,b) incl. at (a,b) itself if $f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$

As in single-var calc, sums, products, quotients (if denom $\neq 0$) of continuous fcns are continuous

BUT, if denom $= 0$, you may/may not be able to make it cont at (a,b)

- $f(x,y) = \dfrac{xy}{x^2+y^2}$ can't make it cont at $(0,0)$

- $f(x,y) = \begin{cases} \dfrac{y^4}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \end{cases}$

- $$f(x,y) = \begin{cases} \dfrac{y^4}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

is continuous.

Symbols

Tuesday, February 2, 2021 1:59 PM

$\forall$ for all

$\exists$ there exists

$\prec$ idea of smaller / less than

$\subset$ subset (strict)

$\cup$ union

$\cap$ intersection

$\therefore$ therefore

$\prod$ product

Σ sum

$\Rightarrow$ implies

$\Leftrightarrow$ IFF

Common Greek / latin letters

δ delta Δ Delta χ Chi

γ gamma

θ theta Θ Theta

ζ

ϕ phi Φ Phi

η nu

σ sigma Σ Sigma

μ mu

ε epsilon E Epsilon

ρ rho P Rho

ω omega Ω Omega

ω omega Ω Omega

λ lambda Λ Lambda

Reminders of Formulas from single var

Generally

$$\frac{d}{dx} ax^3 = 3ax^2$$

We're thinking of "a" as a const but you could think of it as a var

→ You could call it y

$$\frac{d}{dx}(yx^3) = 3yx^2$$

Still y (or a) is still a const bc we're diff'ing wrt x

But you can diff wrt y:

$$\frac{d}{dy}(yx^3) = x^3$$

→ ie, if you plug in some const for x, then the eqn true

eg $x = -1$

$$\frac{d}{dy}(-y) = -1 = (-1)^3$$

eg $\frac{d}{dy}(8y) = 8$

If you want to emphasize that x is const when you diff wrt y, you could call it "a" and write

$$\frac{d}{dy}(ya^3) = a^3$$

Another Formula

$$\frac{d}{dx} e^{ax} = ae^{ax}$$

equivalently:

$$\frac{d}{dx}(e^{yx}) = \frac{d}{dx}(e^{xy}) = ye^{xy}$$

$$\frac{d}{dy}(e^{xy}) = xe^{xy}$$

In MV calc, when you have multiple vars & diff wrt one of them, we write ∂ instead of d.

So

$$\frac{\partial}{\partial x} e^{xy} = ye^{xy}$$

In general, if $f(x,y)$ is a fcn of 2 vars, then

$\frac{\partial f}{\partial x}$ is what you get if you treat y as a const and take a deriv. wrt x

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{y=100} = \frac{d}{dx} \underbrace{f(x,100)}_{\text{single var fcn}}$$

eg $f(x,y) = x\sin(y) + e^x + y$

$$\frac{\partial f}{\partial x} = \sin(y) + e^x$$

NOT EQUAL !

$$\frac{\partial f}{\partial y} = x\cos(y) + 1$$

$$\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial y}$$

"partial derivatives of f"

In 3 variables $f(x,y,z)$

then we have: $\frac{\partial f}{\partial x}, \frac{\partial f}{dy}, \frac{\partial f}{dz}$

eg $f(x,y,z) = xyz$

$$\frac{\partial f}{\partial x} = yz \qquad \frac{\partial f}{\partial y} = xz \qquad \frac{\partial f}{\partial z} = xy$$

What if we differentiate mult times?

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(yz) = 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(xz) = z$$

equal !

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(yz) = z$$

True in general

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$ as long as the partial derivatives are continuous

In general, we only work w fcns whose nth derivatives exist and are continuous

Caveat: sometimes $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are def'd but not continuous & various properties fails

$D_x f$

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$ $\quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$

Exercise $x \sin y + e^x + y$

$\frac{\partial f}{\partial x} \left(\frac{\partial y}{\partial x} \right.$

$= \cos y$

Properties

- sum rule: $\frac{\partial (f+g)}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}$
- scalar mult: $\frac{\partial (af)}{\partial x} = a \frac{\partial f}{\partial x}$, $a \in \mathbb{R}$

  similarly: $\frac{\partial (yf)}{\partial x} = y \frac{\partial f}{\partial x}$
- Product rule: $\frac{\partial}{\partial x}(f \cdot g) = \frac{\partial f}{\partial z} \cdot g + f \cdot \frac{\partial g}{\partial z}$

  (same if we mix up x, y, z)

Let $(a,b) \in D \subseteq \mathbb{R}^2$ and f def'd & diff'able on D.

Consider $g(t) = f(a, b+t)$

this is a one-var fcn

$$\frac{dg}{dt} = \frac{\partial f}{\partial y}(a, b+t)$$

Intuitively Why is $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$?

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\partial}{\partial x}\left(\frac{f(x,y+h)-f(x,y)}{h}\right)\right)$$

assume we can interchange derivative and lim

$$\frac{d}{dx}\left((a+h)x^3\right) = 3(a+h)x^2$$

Notice h & y const wrt x

SO

$$\frac{\partial}{\partial x}\left(\frac{f(x,y+h)-f(x,y)}{h}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{f(x,y+h)-f(x,y)}{h}\right)$$

$$= \frac{\frac{\partial}{\partial x}(f(x,y+h)) - \frac{\partial}{\partial x}(f(x,y))}{h}$$

$$= \frac{\frac{\partial f}{\partial x}(x,y+h) - \frac{\partial f}{\partial x}(x,y)}{h}$$

so $$\frac{\partial^2 f}{\partial x\,\partial y} = \lim_{h\to 0}\left(\frac{\frac{\partial f}{\partial x}(x,y+h)-\frac{\partial f}{\partial x}(x,y)}{h}\right)$$

set $g = \frac{\partial f}{\partial x}$

$$= \lim_{h\to 0}\left(\frac{g(x,y+h)-g(x,y)}{h}\right)$$

Tom Apostol Calc II proves everything rigorously

$$= \frac{\partial g}{\partial x}$$

$$= \frac{\partial(\partial f/\partial x)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$$

2.3 Tanget Planes

Reminder on tangent lines

$$y = f(x)$$

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$\Rightarrow f(x_1) - f(x_0) \approx (x_1 - x_0)\frac{df}{dx}(x_0)$$

$$\Rightarrow \boxed{f(x_1) \simeq f(x_0) + \frac{df}{dx}(x_0)\cdot\Delta x}$$

"linear appoximation"

bc — if fix x_0 & let x_1 vary, then:

approximation

bc — if fix x_0 & let x_1 vary, then:

$f(x_0) + \Delta x \cdot \frac{df}{dx}(x_0)$

$= f(x_0) + \frac{df}{dx}(x_0)\cdot(x_1 - x_0)$

is a linear func of x_1 that's "approx" $f(x_1)$

Approximation is best when x_1 is close to x_0

In other words, the line:

$y = f(x_0) + \frac{df}{dx}(x_0)\cdot(x_1 - x_0)$

where x_0 is fixed & x_1 varies

is best linear approximation to f near x_0

aka tan line @ x_0

Idea given $f(x,y)$ and (x_0, y_0) in its domain, then the tan plane should be given by the linear fcn of x & y that best approx. $f(x,y)$ near (x_0, y_0)

Say (x_1, y_1) is near (x_0, y_0).

$f(x_1, y_1) \approx f(x_0, y_0) + \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}$

$\approx \underbrace{f(x_0, y_0) + (x_1 - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y_1 - y_0)\frac{\partial f}{\partial y}(x_0, y_0)}_{\text{this is a linear fcn of } x_1 \text{ \& } y_1}$

the Func

$z = f(x_0, y_0) + (x_1 - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y_1 - y_0)\frac{\partial f}{\partial y}(x_0, y_0)$

is a linear fcn of x_1, y_1 (aka x, y) that is a good approx to $f(x_1, y_1)$ when (x_1, y_1) is near (x_0, y_0)

Notice $z = f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0)$

def's a plane in $\mathbb{R}^3$

↳ it's the tan plane to $z = f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$

find tangent plane at $(x_0, y_0) = (1, 2)$

Recall $\frac{\partial f}{\partial x} = ye^{xy}$

$\frac{\partial f}{\ }$ $\quad {}^{xy}$

∂x

$$\frac{\partial f}{\partial y} = x e^{xy}$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(1,2)$$

$$= 2e^{(1)(2)} = 2e^2$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(1,2) = e^2$$

$$f(x_0, y_0) = e^2$$

$$\left.\begin{aligned} z &= e^2 + (x-1)(2e^2) + (y-2)e^2 \\ &= 2e^2 x + e^2 y + e^2 - 2e^2 - 2e^2 \\ &= 2e^2 x + e^2 y - 3e^2 \\ &= e^2(2x + y - 3) \end{aligned}\right\}$$

eq. of the tangent plane

2.4 Directional derivatives and Gradient

Tuesday, February 16, 2021 12:11 PM

Definition. For a vector $\vec{v}$ and a function f defined on a domain D containing (a,b)

We define $D_{\vec{v}} f(a,b) =$

$$\lim_{h \to 0} \frac{f((a,b) + h\vec{v}) - f(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{f(a + hv_1, b + hv_2) - f(a,b)}{h}$$

"rate of change of f in the direction $\vec{v}$"

Notice

$$\frac{\partial f}{\partial x} = D_{\hat{\imath}} f \qquad \frac{\partial f}{\partial y} = D_{\hat{\jmath}} f \qquad \text{in 3D } \frac{\partial f}{\partial z} = D_{\vec{k}} f$$

What about $D_{\vec{v}} f$ for:

- $\vec{v} = 2\hat{\imath}$

$$\lim_{h \to 0} \frac{f(a + 2h, b) - f(a,b)}{h}$$

$h' = 2h$

$$\hookrightarrow \lim_{h' \to 0} \frac{f(a + h', b) - f(a,b)}{h'/2}$$

$$= \lim_{h' \to 0} 2 \cdot \frac{f(a + h', b) - f(a,b)}{h'}$$

$$= 2 \lim_{h' \to 0} \dots = 2 \frac{\partial f}{\partial x}$$

- $\vec{v} = c\hat{\imath}$, $c \in \mathbb{R}$

then $D_{\vec{v}} f = c \dfrac{\partial f}{\partial x}$

$$D_{c\vec{v}} f = c D_{\vec{v}} f$$

- Notice $D_{2\vec{v}} = 2 D_{\vec{v}} f$ can be written as.

$$D_{\vec{v} + \vec{v}} = D_{\vec{v}} f + D_{\vec{v}} f$$

Conjecture

$$D_{\vec{v} + \vec{w}} f = D_{\vec{v}} f + D_{\vec{w}} f$$

Equivalently

Say $\vec{v} = v_1 \vec{\imath} + v_2 \vec{\jmath}$

$$D_{\vec{v}} f = D_{v_1 \vec{\imath}} f + D_{v_2 \vec{\jmath}} f$$

$$= v_1 D_{\vec{\imath}} f + v_2 D_{\vec{\jmath}} f$$

$$= v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}$$

$$= \vec{v} \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

In fact, if $D_{\vec{v}} f = v_1 \partial f \quad , v_2 \partial f$

$$= \vec{v} \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

In fact, if $D_{\vec{v}} f = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}$ and $D_{\vec{w}} f = w_1 \frac{\partial f}{\partial x} + w_2 \frac{\partial f}{\partial y}$ then the conjecture $D_{\vec{v}+\vec{w}} f = D_{\vec{v}} f + D_{\vec{w}} f$ is true

Let's prove $D_{\vec{v}} f = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}$

$$D_{\vec{v}} f = \lim_{h \to 0} \frac{f(a+hv_1, b+hv_2) - f(a,b)}{h}$$

Idea: going from $(a,b) \to (a+hv_1, b+hv_2)$ can be done in steps $(a,b) \to (a+hv_1, b)$
$\to (a+hv_1, b+hv_2)$

$$= \lim_{h \to 0} \frac{[f(a+hv_1, b+hv_2) - f(a+hv_1, b)] + [f(a+hv_1, b) - f(a,b)]}{h}$$

$$= \lim_{h \to 0} \frac{f(a+hv_1, b+hv_2) - f(a+hv_1, b)}{h} + \lim_{h \to 0} \frac{f(a+hv_1, b) - f(a,b)}{h}$$

Let's do each separately:

$$\lim_{h \to 0} \frac{f(a+hv_1, b) - f(a,b)}{h} = D_{(v_1, 0)}$$

$$= D_{v_1 \vec{i}} f = v_1 D_{\vec{i}} f = v_1 \frac{\partial f}{\partial x}$$

$$\lim_{h \to 0} \frac{f(a+hv_1, b+hv_2) - f(a+hv_1, b)}{h}$$

glossing over some technicalities

$$= \lim_{h \to 0} D_{(0,v_2)} f(x+hv_1, y)$$

$$= \lim_{h \to 0} v_2 \frac{\partial f}{\partial y}(a+hv_1, b)$$

$$= v_2 \frac{\partial f}{\partial y}(a,b)$$

if $\frac{\partial f}{\partial y}$ is continous

Conclusion If f is differentiable at and near (a,b) and the partial derivative are continuous near (a,b) then

$$D_{\vec{v}} f(a,b) = v_1 \frac{\partial f}{\partial x}(a,b) + v_2 \frac{\partial f}{\partial y}(a,b)$$

$\Rightarrow$ Conjecture is true

Conclusion: $D_{\vec{v}} f = \vec{v} \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is called gradient of f and denoted ∇f

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is called gradient of f and denoted ∇f

In 3D: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

In general: $D_{\vec{v}} f = \vec{v} \cdot \nabla f$ as long as partial derivatives of f are continuous (near the point in question)

Formal def of near:
we say that P happens/is true "near (a,b)" if $\exists\, \varepsilon > 0$ such that P is true for all (x,y) such that $\|(x,y) - (a,b)\| < \varepsilon$
↳ same idea in $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^n$ etc
aka – "happens in a neighborhood of"

Terminology f is "continuously differentiable in a region D" → aka "smooth"
if the partial derivatives of f exist and are continuous in D.

Geometric Meaning of Gradient

Say we want to compare $D_{\vec{v}} f$ for different $\vec{v}$

Note: $D_{\vec{v}} f = \vec{v} \cdot \nabla f$

Let's fix $\|\vec{v}\|$ and vary the direction

- $D_{\vec{v}} f$ is 0 when $\vec{v} \perp \nabla f$
- $D_{\vec{v}} f$ is biggest when $\vec{v}$ is in same direction as ∇f
- $D_{\vec{v}} f$ is smallest when $\vec{v}$ is in opposite dir as ∇f

ex If f describes temp and you're cold, then you want to go in direction of ∇f

↳ hot? → go in dir of $-\nabla f$

- level curves are always $\perp \nabla f$
  e.g. topographical map
  ∇ elevation $\perp$ lines

Multidimensional Linear Functions

Friday, February 26, 2021 8:12 AM

Vectors in n-dimensions are the set $\mathbb{R}^n$ = set n-tuples of real #s.
$\mathbb{R}$ = set of scalars

Write
$\vec{x} = (x_1, x_2, \dots, x_n)$
$\vec{v} = (v_1, v_2, \dots, v_n)$

Key Operations
① Addition for $\vec{x}, \vec{y} \in \mathbb{R}$
$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$
② Scalar Mult for $a \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}$
$a\vec{x} = (ax_1, ax_2, \dots, ax_n)$
③ Dot prod $\vec{x}, \vec{y} \in \mathbb{R}$
$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots x_n y_n$
$= \sum_{i=1}^{n} x_i y_i$

- Dot product is bilinear (distributive), $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2$, is positive if $\vec{x} \neq \vec{0}$

Cauchy-Schwarz
$|\vec{x} \cdot \vec{y}| \le \|\vec{x}\| \cdot \|\vec{y}\|$

Triangle Inequality
$\|\vec{x} + y\| \le \|\vec{x}\| + \|\vec{y}\|$

Coordinates
- $x_1, x_2, \dots, x_n$ are coords of $\vec{x}$
- x_i = ith coord = $\vec{e}_i \cdot \vec{x}$
  where e_i is the ith coord vector
  $\vec{e}_1 = (1, 0, 0, \dots, 0)$
- $\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i$
  → every vector is a linear combo of the coord vectors $\vec{e}_i$

Linear Functions

e.g. $\mathbb{R}^4 \to \mathbb{R}^7$ $\mathbb{R}^7 \to \mathbb{R}^2$ etc

Idea: the derivative of f at $(x_0, y_0 = f(x_0))$ is given by the linear fcn that best approximates f near (x_0, y_0)
i.e. $y = f'(x_0)(x - x_0) + y_0$
- multiplication by $(f'(x_0))$ is linear part of the fcn
- the $(-x_0)$ and $(+y_0)$ are translations
- in general, when you combine translation w/a linear fcn, you get an affine fcn
  → technically $[y = f'(x_0)(x - x_0) + y_0]$ is affine and the linear part is mult by $[f'(x_0)]$

affine: $2x + 3 = y$
linear: $2x = y$ (this is also affine)

In two dimensions
- given $z = f(x,y)$, the best affine fcn that approximates f near $(x_0, y_0, z_0 = f(x_0, y_0))$ is:

$$z = \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + z_0$$

$a := \frac{\partial f}{\partial x}(x_0, y_0)$ $b := \frac{\partial f}{\partial y}(x_0, y_0)$

$$z = \underbrace{(ax+by)}_{\text{linear part}} + \underbrace{(z_0 - ax_0 - by_0)}_{\text{translation}}$$

affine

Note: the translation is just to ensure that the "linear approximation to f" goes through the point (x_0, y_0, z_0)

The derivative is contained in the linear part

Examples of Linear Fcns

$x \mapsto ax$ ("maps to") — the function sending input x to output ax

$\mathbb{R}^1 \to \mathbb{R}^1$

"source" "domain" $\longrightarrow$ "range of possible values" "codomain"

$(x, y) \longmapsto ax + by$ — source is $\mathbb{R}^2$, target domain is $\mathbb{R}^1$; input elements of $\mathbb{R}^2$ and outputs are in $\mathbb{R}^1$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^1$

$(x, y, z) \longmapsto ax + by + cz$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^1$

$(x, y) \longmapsto (ax+by, cx+dy)$

$\mathbb{R}^2 \to \mathbb{R}^2$

Linear Fcns

Def

a function $f: \underset{\text{source}}{\mathbb{R}^n} \longrightarrow \underset{\text{target}}{\mathbb{R}^p}$ is linear if

1. for $\vec{x}, \vec{y} \in \mathbb{R}^n$

   $f(\underbrace{\vec{x} + \vec{y}}_{\text{addition in } \mathbb{R}^n}) = f(\vec{x}) \underbrace{+}_{\text{addition in } \mathbb{R}^p} f(\vec{y})$

2. for $\vec{x} \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

   $f(\underbrace{a\vec{x}}_{\text{scalar mult in } \mathbb{R}^n}) = \underbrace{a f}_{\text{scalar mult in } \mathbb{R}^p}(\vec{x})$

Conclusions (what happens if f is linear)

- For $a, b, \vec{x}, \vec{y}$ we have:

  $f(a\vec{x} + b\vec{y}) = f(a\vec{x}) + f(b\vec{y})$

  $= af(\vec{x}) + bf(\vec{y})$ $\longrightarrow$ respects linear combos

- For any positive integer m and $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n \in \mathbb{R}^n$ and $a_1, a_2, \ldots, a_m \in \mathbb{R}$

  $f(a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_m\vec{x}_m) = f\left(\sum_{i=1}^{m} a_i\vec{x}_i\right)$

  $= a_1 f(\vec{x}_1) + a_2 f(\vec{x}_2) + \cdots + a_m f(\vec{x}_m)$

  $= \sum_{i=1}^{m} a_i f(\vec{x}_i)$

Q/ Given f and $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ how do we find $f(\vec{v})$? (in terms of coords of $\vec{v}$)

A/ $f(\vec{v}) = f\left(\sum_{i=1}^{n} v_i \vec{e}_i\right)$

$$= \sum_{i=1}^{n} v_i F(\vec{e}_i)$$

So if we know $v_1, \dots, v_n$ and $F(\vec{e}_1), F(\vec{e}_2), \dots, F(\vec{e}_n)$, then we can find $F(\vec{v})$

Recall

each $F(e_i)$ is a vector in $\mathbb{R}^p$

Idea: F is specified by a collection of n vectors in $\mathbb{R}^p$

→ ie, if you know those n vectors and you know that F is linear, then you know F.

In fact, given any collection n vectors in $\mathbb{R}^p$ (call them $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n \in \mathbb{R}^p$) then we can find a linear fcn:

$F: \mathbb{R}^n \to \mathbb{R}^p$

s.t. $F(\vec{e}_i) = \vec{x}_i$ for $i = 1, \dots, n$

→ This tells us there is a one-to-one correspondence (bijection) between

linear fcns from $\mathbb{R}^n$ to $\mathbb{R}^p$	AND	collection of n vectors in $\mathbb{R}^p$

In terms of coords

$\vec{x}_1 = F(\vec{e}_1) = (a_{11}, a_{21}, \dots, a_{p1})$

$\vec{x}_j = F(\vec{e}_j) = (a_{1j}, a_{2j}, \dots, a_{pj})$

↳ we assoc. the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{bmatrix}$$ → represents the linear fcn F

- each $F(\vec{e}_j)$ is a column vector in this matrix
- matrix has p rows & n columns

Rows

correspond to coords of target $\mathbb{R}^p$

Columns

correspond to coords of domain $\mathbb{R}^n$

Q/ Given M, how to evaluate $F(\vec{v})$ for $\vec{v} = (v_1, \dots, v_n)$?

A/ We can derive a formula using the fact that F is linear

$$F(\vec{v}) = \sum_{j=1}^{n} v_j F(\vec{e}_j)$$

$$= \sum_{j=1}^{n} v_j (a_{1j}, a_{2j}, \dots, a_{pj})$$

$$= \sum_{j=1}^{n} (v_j a_{1j}, v_j a_{2j}, \dots, v_j a_{pj})$$

$$= \sum_{j=1}^{n} v_j a_{1j}, \sum_{j=1}^{n} v_j a_{2j}, \dots \sum_{j=1}^{n} v_j a_{pj}$$

Conclusion

the ith coord of $f(\vec{v})$ is $\sum_{j=1}^{n} a_{ij} v_j$

There's a natural 1-1 correspondence btw linear fns $\mathbb{R}^n$ to $\mathbb{R}^p$ and $p \times n$ matrices w/ real coefficients.

...his is $M \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ (matrix mult)

Last Lecture Review

Recall a linear function $f: \mathbb{R}^n \to \mathbb{R}^p$ is a fcn satisfying

$$f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y}) \quad \forall\ \vec{x},\vec{y} \in \mathbb{R}^n$$
$$f(a\vec{x}) = af(\vec{x}) \quad a \in \mathbb{R}$$

There's a natural 1-1 correspondence btw linear fcs $\mathbb{R}^n$ to $\mathbb{R}^p$ and $p \times n$ matrices w/ real coefficients.

What is correspondence?

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & \ddots & \\ \vdots & & & \\ a_{p1} & & & a_{pn} \end{pmatrix}$$ corresponds to f

if any of the following equivalent conditions are true:

- $f(e_j) = (a_{1j}, a_{2j}, \dots, a_{pj})$ for $j = 1, \dots, n$
- jth column of A is $f(e_j)$ (viewed as a column vector)
- $f(x_1, x_2, \dots, x_n)$ has ith component $\sum_{j=1}^{n} a_{ij} x_j \quad i = 1, \dots, p$

If $f_1, f_2, \dots, f_p$ denote component vectors in $\mathbb{R}^p$, then

$$f\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{p} a_{ij} x_j f_i\right)$$
$$= \sum_{i=1}^{p}\left[\sum_{j=1}^{n} a_{ij} x_j\right] f_i$$

$$f(x_1, \dots, x_n) = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Now what if we compose fcns?

say

$f: \mathbb{R}^n \to \mathbb{R}^p$
$g: \mathbb{R}^p \to \mathbb{R}^q$
$g \circ f: \mathbb{R}^n \to \mathbb{R}^q$

sends $x \in \mathbb{R}^n$ to $g(f(\vec{x})) \in \mathbb{R}^q$

Fact: If f and g are linear, then so is $g \circ f$

e.g. $g(f(\vec{x}+\vec{y})) = g(f(\vec{x}) + f(\vec{y}))$
$= g(f(\vec{x})) + g(f(\vec{y}))$

Say f corresponds to matrix A ($p \times n$ matrix) and g corresponds to B ($q \times p$ matrix)

Q/ Then which $q \times n$ matrix corresponds to $g \circ f$?
A/ Matrix product BA

Suppose C corresponds to $g \circ f$. Then, its jth column is $g(f(e_j))$

Q/ What is $g(f(e_j))$ in terms of A & B?

$$g(f(e_j)) = g(j\text{th column of } A)$$
$$= g(a_{1j}, a_{2j}, \dots, a_{pj})$$ a vector of length p

its ith component is $\sum_{k=1}^{p} b_{ik} a_{kj}$

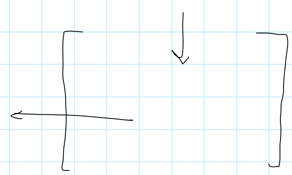
its ith component is $\sum_{k=1}^{p} b_{ik} a_{kj}$

so this is the ij component/coeff of C (def'd as the matrix representing $g \circ f$)

$$\boxed{C = BA}$$

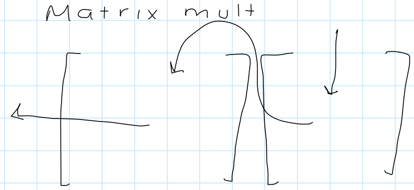
This is matrix multiplication
→ C is the matrix w/columns $Bf(e_j)$
⇒ col(C) correspond to col(A)
⇒ row(C) correspond to row(B)
⇒ col(B) & row(A) just get jumbled around

Thinking in terms of input/output:
col(A) correspond to components of input of f and rows of A correspond to components of the output of f
(similar for B and g)



in through the columns, out through the rows

Matrix mult



Limits & Interior

Open ball

$$B(\vec{a}; r) = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| < r \}$$

Closed ball

$$\overline{B}(\vec{a}; r) = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| \leq r \}$$

Let D be a subset of $\mathbb{R}^n$. We say that $\vec{x} \in D$ is an interior of D if

Intuitively: if $\vec{y}$ is near $\vec{x}$, then $\vec{y} \in D$

formally: $\exists \delta > 0$ s.t $B(\vec{x}; \delta) \subseteq D$

We say D is open if every point if every point of D is an interior point ie, an "open subset of $\mathbb{R}^n$"

Examples
- open interval for $n=1$
- union of open intervals ($n=1$)

union of open intervals ($n=1$)
- all of $\mathbb{R}^n$
- $\emptyset$ the empty set
- open ball $B(a;r)$ (any n) ← inside of a circle
- $n=2$: interior of a square/any polygon
- $n=2$: set of (x,y) satisfying a $\underbrace{\text{strict}}$ linear inequality like:

$x>0$
$x>-7$
$y<3$
$x+y<4$
$ax+by<c$

less than, rather than less than or equal to

- same for linear inequalities for any n

Usually we prefer to consider a fcn def'd on an open domain.
Why?
- If f def'd at $\vec{x}$, then f is def'd near $\vec{x}$ and therefore, we can talk about $\lim\limits_{\vec{y}\to\vec{x}}$ and f will be def'd at $\vec{y}$ near $\vec{x}$

- Another way to state defn of open set:
  D is open if whenever $\vec{x}\in D$, then all points sufficiently close to $\vec{x}$ are also in D.

Nonexample
$D=$ a point not open
and indeed if f is def'd at only a single point, we can't talk about derivatives or limits at that point.

Other non-open sets
- a line in $\mathbb{R}^2$ (or $\mathbb{R}^3$, etc)
- a plane in $\mathbb{R}^3$
- closed ball $\overline{B}(a;r)$ $r\geq 0$
- closed interval
- half-open interval
- square (incl. the boundary) in $\mathbb{R}^2$
- an open square along with a single point on the boundary

Another intuitive def of open
- a set is open if it has no boundary points

Derivatives in Multiple Dimensions

Idea: derivative of a fcn $f: D\longrightarrow\mathbb{R}$ with $D\subseteq\mathbb{R}$ (open) at a point $x_0\in D$ is a number $f'(x_0)$
We should think of it as a 1×1 matrix, ie, as a linear fcn from $\mathbb{R}^1$ to $\mathbb{R}^1$ that approximates f near (x_0,y_0) $y_0=f(x_0)$

Technical Note. If L is linear, then $L(0)=0$,
So if we want to translate L to the point (x_0,y_0), we really consider the affine fcn
$$y=L(x-x_0)+y_0$$
$$=L(x)+y_0-L(x_0)$$

So when we say L approximates f near (x_0,y_0) we really mean $L(x-x_0)+y_0$ approximates f.
($\Leftrightarrow$) L itself approximates $f(x+x_0)-y_0$

This applies to $x_0\in\mathbb{R}^n$, $y_0\in\mathbb{R}^p$ and $L:\mathbb{R}^n\to\mathbb{R}^p$
ie, if we translate L to (x_0,y_0), we take $L(\vec{x}-\vec{x}_0)+\vec{y}_0$.

Suppose $f: D\to\mathbb{R}^p$ where D is an open subset of $\mathbb{R}^n$

For $\vec{x}_0 \in D$, we want to define what we mean by $f'(x_0)$.
It will be a linear fcn from $\mathbb{R}^n$ to $\mathbb{R}^p$.

s.t.

$$\underbrace{f'(x_0)}_{\text{linear fcn}}\underbrace{(\vec{x} - \vec{x}_0)}_{\text{input to linear fcn}} + f(\vec{x}_0)$$

is the best affine approximation to f near $\vec{x}_0$

<u>More precisely</u> we define "best" and what it means for f to be diff'able

f is diff'able at $\vec{x}_0$ iff $f'(\vec{x}_0)$ exists

<u>Even more precisely</u>

<u>Recall</u> precise def for $n = p = 1$

<u>Attempt to generalize to arbitrary p, n</u>

$$f'(\vec{x}_0) = \lim_{\vec{x} \to \vec{x}_0} \frac{f(\vec{x}) - f(\vec{x}_0)}{\vec{x} - \vec{x}_0}$$

<u>PROBLEM</u>

can't divide vectors by other vectors

Instead, the gen def will be:

$$f(\vec{x}) - f(\vec{x}_0) \approx f'(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

for $\vec{x}$ near $\vec{x}_0$

Q/ what do we mean precisely by "$\approx$"?

A/ ~~Look at~~

Notice that the eq of $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$

is equivalent to

$$0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0}$$

<u>Key Fact</u>

a limit approaches 0 iff its magnitude approaches 0.

So, def of derivative is equivalent to:

$$0 = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} \right|$$

$$= \lim_{x \to x_0} \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|}$$

So

$$|x - x_0|$$

so if we replace x, x_0 with $\vec{x}, \vec{x}_0$, then we get.

$$0 = \lim_{\vec{x} \to \vec{x}_0} \frac{\| F(\vec{x}) - F(\vec{x}_0) - F'(\vec{x}_0)(\vec{x} - \vec{x}_0) \|}{\| \vec{x} - \vec{x}_0 \|}$$

No more division by vectors !

use this as def of $F'(\vec{x}_0)$

Critical Points, Implicit Function Theorem

Friday, February 26, 2021 8:16 AM

Last Lecture

Single-variable

say f is def'd on an open domain $D \subseteq \mathbb{R}$, $x_0 \in D$

Two definitions of derivatives:

① $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

diff'able at x_0 iff this limit exists

② We say L is the derivative of f at x_0 if

$$0 = \lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - hL|}{|h|}$$

We say f is diff'able at x_0 iff such an $L \in \mathbb{R}$

Note: if such an L exists, it's unique.

$f'(x_0) := L$ if L exists

Generalization to multiple dimensions

say $F: D \to \mathbb{R}^p$ with $D \subseteq \mathbb{R}^n$ an open domain and $\vec{x}_0 \in D$

we say F is diff'able at $\vec{x}_0$ if there's a linear map $L: \mathbb{R}^n \to \mathbb{R}^p$ s.t.

$$\lim_{\vec{h} \to 0} \frac{\| F(\vec{x}_0 + \vec{h}) - F(\vec{x}_0) - L(\vec{h}) \|}{\|\vec{h}\|} = 0$$

Recall: $\vec{h}, \vec{x}_0 \in \mathbb{R}^n$

$F(\underbrace{-}_{\text{anything}}), L(\vec{h}) \in \mathbb{R}^p$

Q/ How do we compute L?

Q/ eg. given

$$F(x,y) = \left(3\cos(xy) - ye^x,\ xy^2 + \frac{y}{x^2+1}\right)$$

$$\mathbb{R}^2 \to \mathbb{R}^2$$

what 2×2 matrix L?

A/ Partial derivatives!

Recall: $\lim_{h \to 0}$ means $\vec{h}$ can approach 0 from any direction and the limit should be the same.

Let's assume F is diff'able and compute the lim from a particular direction and $L = F'(\vec{x}_0)$

Say $x_1, \dots, x_n$ are coords on $\mathbb{R}^n$

Consider: $\vec{h} = (h, 0, 0, \dots, 0) \in \mathbb{R}^n$ with $h \in \mathbb{R}$

ie, $\vec{h}$ approaches 0 along x_1-axis

So $\|\vec{h}\| = h$

Write: $\vec{x}_0 = (s_1, s_2, \dots, s_n)$ and

$$F(\vec{x}_0 + \vec{h}) - F(\vec{x}_0) - L(\vec{h}) = F(s_1+h, s_2, \dots, s_n) - F(s_1, s_2, \dots, s_n) - L(h\vec{e}_1)$$

because $\vec{h} = h\vec{e}_1$, $\vec{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$

$$= F(s_1+h, s_2, \dots, s_n) - F(s_1, \dots, s_n) - hL(\vec{e}_1)$$

because L is linear

Recall saying that $L = f'(\vec{x}_0)$ is, by definition, say that:

$$0 = \lim_{\vec{h} \to 0} \frac{\| f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - L(\vec{h}) \|}{\|\vec{h}\|}$$

$$= \lim_{\vec{h} \to 0} \frac{\| f(s_1 + h, s_2, \dots, s_n) - f(s_1, s_2, \dots, s_n) - hL(\vec{e}_1) \|}{h}$$

use $\|a\vec{v}\| = a\|\vec{v}\|$

looks like partial derivative

$$= \lim_{\vec{h} \to 0} \left\| \frac{f(s_1 + h, s_2, \dots, s_n) - f(s_1, s_2, \dots, s_n)}{h} - L(\vec{e}_1) \right\|$$

first column of L matrix

so, this limit is 0

In other words, as $h \to 0$, $\frac{f(s_1 + h, s_2, \dots, s_n)}{h}$ approaches $L(\vec{e}_1)$

$$\Rightarrow L(\vec{e}_1) = \lim_{h \to 0} \frac{f(s_1 + h, s_2, \dots, s_n) - f(s_1, \dots, s_n)}{h}$$

$$= \frac{\partial f}{\partial x_1}$$

Q/ We talked about partial derivatives of fcns w/ multiple inputs but one output. What does $\frac{\partial f}{\partial x_1}$ mean if $f : D \to \mathbb{R}^p$ and $p > 1$?

A/ Apply $\frac{\partial}{\partial x_1}$ to each of the p components

Explicitly: If

$f(x_1, \dots, x_n) = (u_1(x_1, \dots, x_n), u_2(x_1, \dots, x_n), \dots, u_p(x_1, \dots, x_n))$

eg. a fcn $f : \mathbb{R}^3 \to \mathbb{R}^2$ is the same as pair of fcns u_1 and u_2, each from $\mathbb{R}^3$ to $\mathbb{R}^1$

then

$$\frac{\partial f}{\partial x_1} = \begin{bmatrix} \partial u_1 / \partial x_1 \\ \partial u_2 / \partial x_1 \\ \vdots \\ \partial u_p / \partial x_1 \end{bmatrix}$$

Conclusion

If f is diff'able at $\vec{x}_0$ and $L = f'(\vec{x}_0)$ then

$$L(\vec{e}_1) = 1^{st} \text{ column of } L$$

$$= \frac{\partial f}{\partial x_1}$$

In general, let $1 \le j \le n$, $L(e_j) = j^{th}$ column of L

$$= \frac{\partial f}{\partial x_j}(\vec{x}_0)$$

∂x_j

partial derivative at $\vec{x}_0$

SO

$$L = \begin{bmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \cdots & \partial u_1/\partial x_n \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \cdots & \partial u_2/\partial x_n \\ \vdots & \vdots & & \vdots \\ \partial u_p/\partial x_1 & \partial u_p/\partial x_2 & & \partial u_p/\partial x_n \end{bmatrix}$$

"total derivative" $F'(\vec{x}_0)$

Caveat this says that if L exists, then L is the linear map $\mathbb{R}^n$ to $\mathbb{R}^p$ given by the matrix of partials

Q/ How to show F is diff'able?

A/ see Apostol on BCourses for proof

Thm
If $F: D \to \mathbb{R}^p$ and $D \overset{\text{open}}{\subseteq} \mathbb{R}^n$ and if $\forall\ 1 \le j \le n$,

$\frac{\partial F}{\partial x_j}(\vec{x})$ exists and is continuous for all

$\vec{x} \in D$, then F is diff'able at all $\vec{x} \in D$

and then $F'(\vec{x})$ is represented by a matrix whose j^{th} column is $\frac{\partial F}{\partial x_j}(\vec{x})$

Notice: the i^{th} row of $F'(\vec{x})$ is $\nabla u_i(\vec{x})$ viewed as row vector where $F = (u_1, u_2, \ldots, u_p)$

FACT: ∇u_i is a vector of length n

so, the j^{th} column corresponds to x_j and the i^{th} row corresponds to u_i

Example Corollary

$F(x,y) = \left(3\cos(xy) - ye^x,\ xy^2 + \frac{y}{x^2+1} \right)$ is diff'able at all $\vec{x} = (x,y) \in \mathbb{R}^2$

PROOF

$$\frac{\partial F}{\partial x} = \begin{bmatrix} -3y\sin(xy) - ye^x \\ y^2 - \frac{2xy}{(x^2+1)^2} \end{bmatrix}$$

$$\frac{\partial F}{\partial y} = \begin{bmatrix} -3x\sin(xy) - e^x \\ 2xy + \frac{1}{x^2+1} \end{bmatrix}$$

Notice both $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are defined and continuous for all $(x,y) \in \mathbb{R}^2$ so, by the Thm, F is diff'able on all of $\mathbb{R}^2$. ■ QED

<u>Note:</u> if $f: D \to \mathbb{R}^1$ is diff'able at $\vec{x}_0$, then $D_{\vec{u}} f(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}$

This follows from the <u>Chain Rule</u>:

<u>Suppose</u> $f: D \to \mathbb{R}^p$, $D \subseteq \mathbb{R}^n$, $g: E \to \mathbb{R}^q$, $E \subseteq \mathbb{R}^p$

suppose $\vec{x}_0 \in D$, $f(\vec{x}_0) \in D$

f is diff'able at $\vec{x}_0$ and g is diff'able at $f(\vec{x}_0)$

<u>Then</u>

$$\underbrace{(g \circ f)'(\vec{x}_0)}_{\text{linear map from } \mathbb{R}^n \text{ to } \mathbb{R}^q} = \underbrace{g'(f(\vec{x}_0))}_{\text{linear map from } \mathbb{R}^p \text{ to } \mathbb{R}^q} \circ \underbrace{f'(\vec{x}_0)}_{\text{linear map from } \mathbb{R}^n \text{ to } \mathbb{R}^p}$$

Concretely — this says you can compute partials of $g \circ f$ in terms of partials of g and f using matrix mult.

Recall for a fcn $f: D \to \mathbb{R}^p$, $\underbrace{D}_{\text{open}} \subseteq \mathbb{R}^n$ and $\vec{x}_0 \in D$

We say f is diff'able at $\vec{x}_0$ if $\exists$ a linear map: $f'(\vec{x}_0): \mathbb{R}^n \to \mathbb{R}^p$ such that:

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\| f(\vec{x}) - f'(\vec{x}_0)(\vec{x} - \vec{x}_0) - f(\vec{x}_0) \|}{\| \vec{x} - \vec{x}_0 \|} = 0$$

aka $f(\vec{x}) \approx f'(\vec{x}_0)(\vec{x} - \vec{x}_0) + f(\vec{x}_0)$ for $\vec{x} \approx \vec{x}_0$

Let $$e(x) := \frac{f(\vec{x}) - f'(\vec{x}_0)(\vec{x} - \vec{x}_0) - f(\vec{x}_0)}{\| \vec{x} - \vec{x}_0 \|}$$

Note: $e(\vec{x})$ is a vector in $\mathbb{R}^p$

so:
$$f(\vec{x}) = f'(\vec{x}_0)(\vec{x} - \vec{x}_0) + f(\vec{x}_0) + \| \vec{x} - \vec{x}_0 \| e(\vec{x})$$

To say $f'(\vec{x}_0)$ is the derivative is equivalent to:
$$\lim_{\vec{x} \to \vec{x}_0} e(\vec{x}) = \vec{0} \in \mathbb{R}^p$$

This makes precise what we mean by "good linear approximation to f near $\vec{x}_0$"

Recall
- If f differentiable a $\vec{x}_0$ then all np partials exist then all np partials exist and $f'(x_0)$ is represented by the matrix of partials
- If all partials exist and are continuous in a nbhd of $\vec{x}_0$ then f is diff'able at $\vec{x}_0$
- In anomolous cases, the partials might exist but f is not diff'able at $\vec{x}_0$

Thm Chain Rule:

If $f: D_1 \to \mathbb{R}^p$, $g: D_2 \to \mathbb{R}^q$, $D_1 \subseteq \mathbb{R}^n$, $D_2 \subseteq \mathbb{R}^p$, $\vec{x}_0 \in D_1$ and $f(\vec{x}_0) \in D_2$ and f diff'able at $\vec{x}_0$ and g diff'able at $\vec{x}_0$ and

$$(g \circ f)'(\vec{x}_0) = g'(f(\vec{x}_0)) \underset{\substack{\uparrow \\ \text{mat mul}}}{\,} f'(\vec{x}_0)$$

Proof sketch

say $f(\vec{x}) = f'(\vec{x}_0)(\vec{x} - \vec{x}_0) + f(\vec{x}_0) + \| \vec{x} - \vec{x}_0 \| e_1(\vec{x}_0)$

$(\ast)$ $g(\vec{y}) = g'(f(\vec{x}_0))(\vec{y} - f(\vec{x}_0)) + g(f(\vec{x}_0)) + \| \vec{y} - f(\vec{x}_0) \| e_2(\vec{y})$

think: $\vec{y}_0 = f(\vec{x}_0)$

Plug in $\vec{y} = f(\vec{x})$,

then $\underbrace{f(\vec{x})}_{=y} = f'(\vec{x}_0)(\vec{x} - \vec{x}_0) + f(\vec{x}_0) + \| \vec{x} - \vec{x}_0 \| e(x)$

and then plug this into $(\ast)$

$g(f(\vec{x})) = g(\vec{y}) = g'(\vec{y}_0)(f'(\vec{x}_0)(\vec{x} - \vec{x}_0)) + g(\vec{y}_0) +$ error terms involving e_1 and e_2

you'll get a term of the form:

$$g'(\vec{y}_0)(\| \vec{x} - \vec{x}_0 \| e_1(\vec{x})) = \| \vec{x} - \vec{x}_0 \| \underbrace{g'(\vec{y}_0)(e_1(\vec{x}))}_{\text{goes to } 0 \text{ as } \vec{x} \to \vec{x}_0}$$

Conclusion

— To compute, you use matmul but proof of chain rule just uses composition of linear fcns

Note Key case is $n=q=1$
then f is a vector-valued fcn of one input
g is a scalar-valued fcn
so f' is same as in first few weeks
g' is ∇g viewed as a row vector
and $\underbrace{g'(f(\vec{x}_0))}_{\text{row vector}}\underbrace{f'(\vec{x}_0)}_{\text{column vector}}$

$= \nabla g(f(\vec{x}_0)) \underset{\text{dot prod}}{\cdot} f'(\vec{x}_0)$

"Key case" bc you can prove multidim chain rule using this case (once for each of nq coeffs of $(g \circ f)'$)

Note formula for direction derivative
$D_{\vec{v}}\, g = \nabla g \cdot \vec{v}$ is a special case of chain rule
(where $f(t) = t\vec{v} + \vec{x}_0$)

Maxima & Minima

Suppose $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$ and $\vec{x}_0 \in D$

Def We say that f has a:

1. Local max at $\vec{x}_0$ if
   $f(\vec{x}) \leq f(\vec{x}_0)$ $\forall \vec{x}$ near $\vec{x}_0$
   (ie $\exists \varepsilon > 0$ s.t. it's true for all $\|x - \vec{x}_0\| < \varepsilon$)
2. Local min at $\vec{x}_0$ if
   $f(\vec{x}) \geq f(\vec{x}_0)$ $\forall \vec{x}$ near $\vec{x}_0$
3. Global max at $\vec{x}_0$ if
   $f(\vec{x}) \leq f(\vec{x}_0)$ $\forall \vec{x} \in D$
4. Global min at $\vec{x}_0$ if
   $f(\vec{x}) \geq f(\vec{x}_0)$ $\forall \vec{x} \in D$

Note global max/min $\Rightarrow$ local max/min
$\not\Leftarrow$

Note for $n=1$, if local max/min then $f'(x_0) = 0$
Similarly for general n, if f has a local max/min at $\vec{x}_0$ then
$\nabla f(\vec{x}_0) = 0$
Also for $n=1$ sometimes $f'(x_0) = 0$ but f doesn't have a local max
Similarly can have $\nabla f(\vec{x}_0) = 0$ but no local max or min

eg

1. $f(x,y) = x^3 + y^3$ $\quad \nabla f = (3x^2, 3y^2)$ $\quad x_0 = (0,0)$
   but not local max/min $\quad \nabla f(x_0) = (0,0)$
   (2D version of $f(x) = x^3$)
2. $f(x,y) = x^2 - y^2$
   $x_0 = (0,0)$ $\quad \nabla f = (2x, -2y)$ $\quad \nabla f(x_0) = (0,0)$
   $\nabla f = \vec{0}$ but not a local min/max
   ↳ called a saddle point (fundamentally multidim)

Defn If $\nabla f(\vec{x}_0) = \vec{0}$ then we say that $\vec{x}_0$ is a critical point of f

Thus
- any local max/min is a critical point
- above we gave ex of critical pts that weren't max/min

Recall in 1 var, if:
$f''(x_0) > 0 \longrightarrow$ local min
$f''(x_0) < 0 \longrightarrow$ local max
$f''(x_0) = 0 \longrightarrow$ unclear

in multivar:
Define Hessian:

If $\vec{x_0}$ is a critical pt of F,
$F: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$
Then define an $n \times n$ matrix of 2nd derivatives, whose ij-coeff is $\frac{\partial^2 F}{\partial x_i x_j}$

__Notice__ ij-coef equals the ji-coeff
$\Rightarrow$ it's a symmetric matrix

__eg__ $n = 2$, $x_1 = x$, $x_2 = y$ $\Rightarrow$ $\vec{x}_0 = (x, y)$

$$\mathrm{Hess}_{\vec{x_0}}(F) = \begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{bmatrix}$$

$$D = \det(\mathrm{Hess}_{\vec{x_0}}(F))$$

$$= D_{xx}F \; D_{yy}F \longleftarrow (D_{xy}F)^2$$

$\hookrightarrow$ can ask — is this scalar positive/negative

__Two-variable 2nd derivative Test__

① If $D > 0$, then F has a local max or local min at $\vec{x_0}$

② If $D < 0$, then F has a saddle point

③ If $D = 0$, then the test doesn't determine what happens.

__Remark__
- In case $D > 0$, you can tell if local max/min by finding the eigenvalues of the Hessian
  - $\to$ positive eigenvalues: local min
  - $\to$ negative eigenvalues: local max

__Comment on Saddlepoints__
- when F has a local max in one direction and a local min in the other
- NOT like 1-D critical pts that aren't a local max/min
  - $\to$ rather, you have a local max in 1-D and a local min in an orthogonal direction (fundamentally multidim)

__eg__ $F(x, y) = x^2 - y^2$ at $(0, 0)$
__then__ if you fix $y = 0$ and let x vary, then F has a local min at x_0
if you fix $x = 0$ and let y vary then you get a local max at x_0

__eg__ $F(x, y) = xy$
__then__ it's a local min in the direction $\vec{u} = (1,1)$
ie, $D_{\vec{u}} D_{\vec{u}} F > 0$
__but__ local max in direction $\vec{u} = (1, -1)$

__Intro to Inverse Fcn Thm__

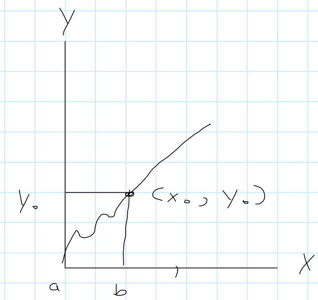
Let $f: [a,b] \to \mathbb{R}$ ... say

Intro to Inverse Fcn Thm

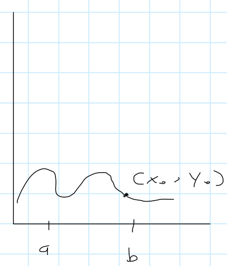
Let $f: [a,b] \longrightarrow \mathbb{R}$ (one-var fcn). say $x_0 \in (a,b)$, say f is diffable
$y_0 = f(x_0)$

Consider 3 cases for $f'(x_0)$. If

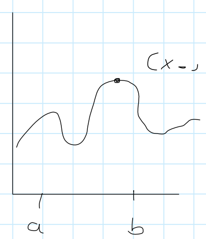
1. $f'(x_0) > 0$



2. $f'(x_0) < 0$



3. $f(x_0) = 0 \rightarrow$ suppose $f''(x_0) < 0$



Suppose we want to inverse the function $f^{-1}(y)$

$$x = f^{-1}(y) \iff y = f(x)$$

Case 1

- say we want $f^{-1}(y_0)$ that should be x_0
- say we want $f^{-1}(y_1)$ have ≥ 2 possibilities for its value

but if y near y_0 and $f'(x_0) \neq 0$, can choose $f^{-1}(y)$ consistently for y near y_0

but not necessarily if $f'(x_0) = 0$

Lagrange multipliers

Thursday, March 4, 2021 12:16 PM

Last time: chain rule, maxima/minima

Upshot

1. critical point iff gradient vanishes
2. local max/min $\Rightarrow$ critical point but not conversely
3. in two-dimensions, can use Hessian determinant:

$$\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2}-\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2$$

$\underbrace{\text{Parametric vs Equation form of a curve in } \mathbb{R}^2}$

A curve, usually denoted by γ is a 1-D subset of $\mathbb{R}^2$

Given by either

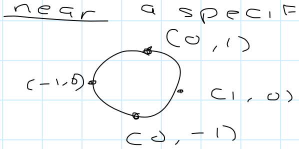
- Parametric: $(x,y) = (x(t), y(t))$
  $t \in \mathbb{R}$
- Equation: $g(x,y)=0$
  then $\gamma = \{(x,y) \in \mathbb{R}^2 \mid g(x,y)=0\}$

eg a circle
parametric $(x,y) = (\cos t, \sin t)$
eqn $x^2+y^2-1=0$

What if we want a parametric form in which one of the variables is the parameter?
ie. $(x,y) = (t, y(t))$
or $(x,y) = (x(t), t)$
In 1st case, eqn form is $y - y(x) = 0$

eg for a circle

- say $x = t$
  get $y = \sqrt{1-x^2}$
  (or $y = -\sqrt{1-x^2}$)
- Can't find single eqn for y in terms of x everywhere (globally)
- but can often find it locally
  ie near a specific point



eg near $(0,1)$ have local parametric form
$y = -\sqrt{1-x^2}$

Q/ but at $(1,0)$ what's the sqrt?

A/ can't solve for x in terms of x near $(1,0)$ or $(-1,0)$ bc the tangent line is vertical
Similarly near $(0,\pm 1)$, can't locally solve for x in terms of y bc the tangent line is horizontal

Implicit Fcn Thm

Implicit Fcn Thm

Suppose $g(x,y)=0$ describes a curve γ and that $a,b \in \gamma$ and (a,b) is not a critical point of g

Then

① If tan line to γ at (a,b) is not vertical then we can solve for y in terms of x near (a,b).

Precisely:

$\Rightarrow$ can find fcn f defined on a nbhd of a (ie some open interval containing a) such that $(x,y) = (t, f(t))$ describes the curve γ near (a,b)

② If the tangent line is not horizontal at (a,b), can solve for x in terms of y near (a,b)

Caveat:

Only works at long as (a,b) is not a critical point of g

Q/ When is tangent line vertical

A/ Recall tangent line is given by

$$\frac{\partial g}{\partial x}(a,b)\cdot(x-a) + \frac{\partial g}{\partial y}(a,b)\cdot(y-b)=0$$

this is vertical if $\frac{\partial g}{\partial y}(a,b)=0$

Better Formulation of Implicit Fcn thm

① If $\frac{\partial g}{\partial y}(a,b)\neq 0$ then can solve for y in terms of x near (a,b)

② If $\frac{\partial g}{\partial x}$ then can solve for x in terms of y near (a,b)

Notice thm automatically doesn't apply if (a,b) is a critical point of g

$\Rightarrow$ we don't have to explicitly require that (a,b) is not critical

Generalizations

- In $\mathbb{R}^3$ consider $g(x_1,x_2,x_3)=0$. This defines a surface (not a curve).
- If at (a_1,a_2,a_3) we have $\frac{\partial g}{\partial x_i}\neq 0$ then can solve for x_i in terms of the other two variables near (a_1,a_2,a_3)
- Similar in $\mathbb{R}^n$. Then $g(x_1,\ldots,x_n)=0$ defines an $(n-1)$ dimensional subset of $\mathbb{R}^n$ and there are $n-1$ parameters.
- What about a curve in $\mathbb{R}^3$?

  Then, need to consider $g(x_1,x_2,x_3)=(g_1,g_3)$

  ie. need to solve $g(x_1,x_2,x_3)=(0,0)$

  for $g: D \to \mathbb{R}^2$, $D \subset \mathbb{R}^3$

  now you have a 2×3 matrix and you consider determinents of 2×2

eg can solve for x_2 and x_3 in terms of x_1 if:

$$\det \begin{vmatrix} \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{vmatrix} \neq 0$$

at the given

use $q \times q$ minors of $q \times n$ matrix

Lagrange Mult

Suppose f a fcn in $\mathbb{R}^2$ and γ is a curve.

Suppose we want to maximize, or minimize $f(x,y)$ among $(x,y) \in \gamma$

How?

if we have a parametrization $(x,y) = (x(t), y(t))$ of γ then its easy using chain rule.

Why? just need to solve $\frac{df}{dt} = 0$

Notice $\frac{df}{dt} = \frac{d f(x(t), y(t))}{dt} = \nabla f \cdot (x'(t), y'(t))$

use chain rule for composition

$$\mathbb{R} \xrightarrow{(x(t),\, y(t))} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

so $\frac{df}{dt} = 0$ precisely when $(x'(t), y'(t))$ aka tangent vector to γ, is $\perp$ to ∇f.

But what if we don't have a parameterization? (if γ given by $g(x,y) = 0$?)

one attempt at an answer.
- if (x,y) isn't a critical point, the implicit fcn thm says there is a parameterization
  BUT doesn't say how to compute it

Lagrange's idea: use ∇g instead of $(x'(t), y'(t))$

How? Tangent line at (a,b) is given by

$$\nabla g(a,b) \cdot [(x,y) - (a,b)] = 0$$

$\Rightarrow$ the tangent vector is $\perp$ to ∇g

ie, for any parameterization $(x(t), y(t))$ of γ we have

$\nabla g \cdot (x'(t), y'(t)) = 0$

therefore, $\nabla f \perp (x'(t), y'(t))$ iff $\nabla f \parallel \nabla g$

in summary

∇g always $\perp$ tangent vector

$\nabla f \perp$ tangent vector whenever $\frac{df}{dt} = 0$

$\nabla f \parallel \nabla g$ when $\frac{df}{dt} = 0$

∇g always $\perp$ tangent vector
∇F $\perp$ tangent vector whenever $\frac{dF}{dt} = 0$
$\nabla F \parallel \nabla g$ when $\frac{dF}{dt} = 0$

→ (and this condition doesn't refer to the parameterization)

When is $\nabla F \parallel \nabla g$?

<u>if:</u> $\nabla F = \lambda \nabla g$ for $\lambda \in \mathbb{R}$

<u>note:</u> $\nabla F = \lambda \nabla g$

$\iff \frac{\partial F}{\partial x} = \lambda \frac{\partial g}{\partial x} \iff \frac{\partial F}{\partial y} = \lambda \frac{\partial g}{\partial y}$

solve these 2 eqns along with the third eqn:
$g(x, y) = 0$
for x, y, λ

<u>Note</u> can do same in 3-dim
→ get 4 eqns for the 4 vars x, y, z, λ
bc $\nabla F, \nabla g \perp$ to tangent plane

Double Integrals

Q given $F(x,y)$ what does it mean to integrate F?

<u>idea:</u> Partial integral wrt one of the variables and consider the other variable as a const

<u>eg</u>

$F(x, y) = x^2 y$

$\int F\,dx = \frac{yx^3}{3} + C$

$\int F\,dy = \frac{x^2 y^2}{2} + C$

gives some notion of indefinite integral

Q/ What about definite integral?

$$\int_1^2 F\,dx = \left.\frac{yx^3}{3}\right|_{x=1}^{x=2} = \frac{y(2)^3}{3} - \frac{y(1)^3}{3} = \frac{7y}{3}$$

<u>Notice:</u> still fcn of y.

<u>Similarly:</u>

$$\int_1^2 F\,dy = \left[x^2 y^2/2\right]_{y=1}^{y=2} = \frac{4x^2}{2} - \frac{x^2}{2} = \frac{3x^2}{2}$$

Q/ How to get # as a definite integral?

A/ Integrate twice, once wrt each var

<u>eg</u>

$$\int_1^2 \left[\int_1^2 F\,dx\right] dy = \int_1^2 \frac{7y}{3}\,dy = \left[\frac{7y^2}{6}\right]_1^2 = \frac{28}{6} - \frac{7}{6} = \frac{21}{6} = \frac{7}{2}$$

Let's try:

$$\int_1^2 \left[\int_1^2 F\,dy\right] dx = \int_1^2 \frac{3x^2}{2}\,dx = \left[\frac{x^3}{2}\right]_{x=1}^{x=2} = \frac{8}{2} - \frac{1}{2} = \frac{7}{2}$$

this is

$$\int_{y=1}^{y=2}\int_{x=1}^{x=2} f\,dx\,dy$$

instead we could go from $x=2$ to $x=3$ but still $y=1$ to $y=2$

then we get.

$$\int_{y=1}^{y=2}\int_{x=2}^{x=3} f\,dx\,dy = \int_{x=2}^{x=3}\left[\frac{3x^2}{2}\right]dx = \left[\frac{x^3}{2}\right] = \frac{27}{2} - \frac{8}{2} = \frac{19}{2}$$

Double Integrals cont., Center of Mass

Monday, March 8, 2021 2:40 AM

Official Reading 3.1, 3.2 of [Co]
Recommended 12.1 of [CH]

Center of mass: 13.1 of [CH], 3.6 of [Co]

Double integrals:

Last time: took $f(x,y)$ and did definite integration twice to get a number

- Subtlety in 2 dimensions: 2 different ways (orders) to integrate:

1st way
- integrate wrt x to get a fcn of y then integrate wrt y to get a #

2nd way
- integrate wrt y first to get a fcn of x, then wrt x.

Miracle: Get same answer $\rightarrow$ part of Fubini's Theorem

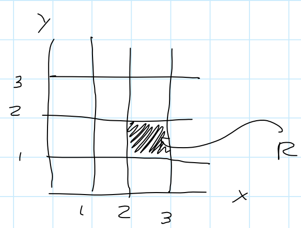
eg. integrate from $x=2$ to $x=3$ & $y=1$ to $y=2$

$\Rightarrow$ integrate over the region:

$$R = [2,3] \times [1,2] = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x \in [2,3] \\ y \in [1,2] \end{matrix} \right\}$$

just like $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

this is a filled in square



eg. $R = [1,3] \times [1,2]$ $\longrightarrow$ this is a rectangle

Notation

$$\int_{y=1}^{y=2} \int_{x=1}^{x=3} f(x,y)\,dx\,dy = \iint_{R=[1,3]\times[1,2] \subseteq \mathbb{R}^2} f(x,y)\,dx\,dy$$

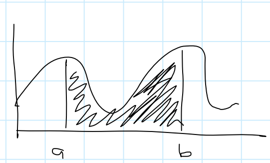
Compare

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx$$

Q/ what does $\iint_R f(x,y)\,dx\,dy$ mean?

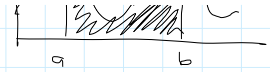
A/ Compare w/ 1 variable:

Consider $\int_{[a,b]} f(x)\,dx$



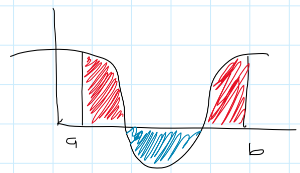
the integral is the

$\int_{[a,b]}$



the integral is the area under the curve

If $f(x) \geq 0$ for $x \in [a,b]$ then it's really the area
But if f goes below the x-axis, then we get signed area

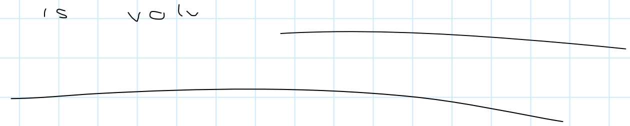


$$\int_{[a,b]} f\,dx = \text{total "signed" area}$$

$= \text{total area of red regions} - \text{area of blue}$

Similarly

$\iint_R f(x,y)\,dx\,dy$ is volu



<u>Note</u> when $f(x,y) < 0$, we count the volume as negative.

- ie, we add all volume truly above the xy plane and subtract all volume below

<u>Note</u> if $f(x,y) = 1$, then $\iint_R f\,dx\,dy = \text{area}(R)$

<u>so</u>

we know how to compute $\iint_R f(x,y)\,dx\,dy$ when R is a rectangle with sides parallel to the x and y axes

Then $R = [a,b] \times [c,d]$
and then

$$\iint_R f\,dx\,dy = \int_c^d \int_a^b f\,dx\,dy = \int_a^b \int_c^d f\,dy\,dx$$

(eg if f is continuous)
or piecewise continuous $\longrightarrow$ $f = \begin{cases} \sim\sim & \text{if } \sim\sim \\ \sim\sim & \text{if } \sim\sim \end{cases}$

<u>Upshot</u> For all the fcns we consider in this class, Fubini's Thm is TRUE

Q/ What about a double integral over a more general region?
<u>Recall</u> suppose that we integrate wrt x first. Then you get a fcn

Q/ What about a double integral over a more general region?

<u>Recall</u> suppose that we integrate wrt x first. Then you get a fcn of y and you integrate wrt y

e.g. $$\int_c^d\int_a^b xy\,dx\,dy = \int_c^d \left.\frac{yx^2}{2}\right]_{x=a}^{x=b} dy$$

$$= \int_c^d y\left(\frac{b^2}{2} - \frac{a^2}{2}\right) dy$$

$$= \left[\frac{y^2}{2}\left(\frac{b^2}{2} - \frac{a^2}{2}\right)\right]_{y=c}^{y=d}$$

Key: this expr has no x's in it (only y and constants)

$$= \frac{(d^2-c^2)(b^2-a^2)}{4}$$

Q/ What if we integrated from $x = \frac{y}{2}$ to $x = y$?

A/ Then $\displaystyle\int_{x=\frac{y}{2}}^{y} f(x,y)\,dy$ would still be a fcn of y (w/o x's). And then when we integrate wrt y, we end up with a const.

<u>Here's How</u>

$$\int_{x=\frac{y}{2}}^{x=y} xy\,dx = \left[\frac{yx^2}{2}\right]_{x=\frac{y}{2}}^{x=y}$$

$$= \frac{y(y^2)}{2} - \frac{y(\frac{y}{2})^2}{2}$$

$$= \frac{y^3}{2} - \frac{y^3}{8} = \frac{3y^3}{8}$$

to compute $\iint$ do:

$$\int_{y=c}^{y=d} \frac{3y^3}{8}\,dy = \left[\frac{3y^4}{32}\right]_{y=c}^{y=d}$$

$$= \frac{3(d^4-c^4)}{32}$$

Q/ What is the geometric interpretation?

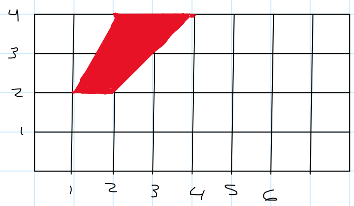
A/ For each y, we go from $x = \frac{y}{2}$ to $x = y$. Then we add up from $y = c$ to $y = d$

<u>e.g.</u> $c = 2$, $d = 4$

At $y = 2$ we go from $x = 1$ to $x = 2$

At $y = 4$ we go from $x = 2$ to $x = 4$

(at $y = 3$ from $x = \frac{3}{2}$ to $x = 3$)



R = red region = $\left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} 2 \le y \le 4 \\ \frac{y}{2} \le x \le y \end{array} \right\}$

is a trapezoid

eg. $$\iint_R xy\,dx\,dy = \frac{3}{32}(d^4 - c^4)$$

$$= \frac{3}{32}(4^4 - 2^4)$$

$$= \frac{3}{2^5}(2^8 - 2^4)$$

$$= \frac{3}{2}(2^3 - 1)$$

$$= \frac{3 \cdot 7}{2} = \frac{21}{2}$$

Q/ Can we do the same integral in the opposite order?

$$\int_2^4 \int_{\frac{y}{2}}^{y} xy\,dx\,dy \overset{?}{=} \int_{\frac{y}{2}}^{y} \int_2^4 xy\,dy\,dx$$

$$= \int_{\frac{y}{2}}^{y} \left. \frac{xy^2}{2} \right]_{y=2}^{y=4} dx$$

$$= \int_{\frac{y}{2}}^{y} 6x\,dx$$

$$= \left. 3x^2 \right]_{\frac{y}{2}}^{y}$$

$$= 3y^2 - \frac{3y^2}{4}$$

→ PROBLEM

this is not a #

For rectangles, we can integrate wrt x or y first.

For region R, it depends on the limits of integration

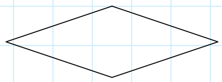
→ If R has 2 vertical sides, you can integrate wrt y first then x

→ If R has 2 horizontal sides, the opposite

Summary

- outer limits of integration must be constants

Q/ What if R is a quadrilateral w/o vertical/horizontal sides?



A/ Break it up into simpler regions and add up the results

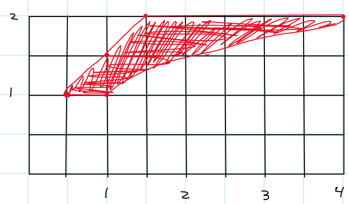
Compare: $\int_a^c f\,dx = \int_a^b f\,dx + \int_b^c f\,dx$

$[a,c] = [a,b] \cup [b,c]$

But first, more examples!

Consider

$$\int_{y=1}^{y=2}\int_{x=\frac{y^2}{2}}^{x=y^2} f\,dx\,dy = \iint_R f\,dx\,dy \quad \text{where } R \text{ is:}$$



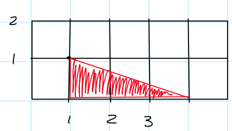
Notice:

non-horizontal sides are curved

Next example: right triangle

hypotenuse: $y = -\frac{x}{2} + \frac{3}{2}$

$x = 3 - 2y$



2 ways to do it

First way: integrate wrt x first

↳ need: two horizontal sides

- bottom horizontal side is the segment $[1,3]$ on x-axis
- top horizontal side is a point $(1,1)$ thought of as a side of length 0
- then if R = the right Δ, then

$$\iint_R f\,dx\,dy = \int_{y=0}^{y=1}\int_{x=1}^{x=3-2y} f\,dx\,dy$$

also think of it as having two vertical sides also think of U as having two vertical sides

→ one side: segment from (1,0) to (1,1)

→ another side: point (3,0)

get

$$\int_{x=1}^{x=3} \int_{y=0}^{y=-\frac{x}{2}+\frac{3}{2}} f \, dy \, dx = \int \qquad \int$$

Change of Variables for Double Integrals

Thursday, March 11, 2021 12:14 PM

R region in the plane — under the graph of $z=f(x,y)$

— if R is a rectangle w/ sides parallel to coord. axes (i.e. $[a,b]\times[c,d]$), then we talked about to compute this

— if R has two horizontal sides (but other sides might not be)

Principle

If $R = R_1 \cup R_2$ and R_1 & R_2 don't overlap other than their boundary, then

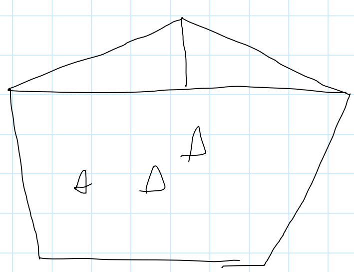
$$\iint_R f(x,y)\,dx\,dy = \int$$

Key: one side of R is parallel to one of the coordinate axis.

More generally: For any triangle, you have two options:

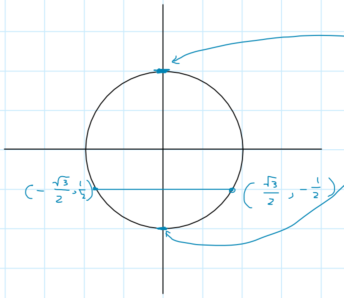
1. rotate so that one side's parallel (technically uses change of variables)
2. break up any triangle into pieces that have one side parallel to one of the coordinate axes

can do something similar if R is a polygon



Q/ What about integrating over the circle $R = \{(x,y)\in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$

A/ can pretend R has 2 horizontal sides



horizontal sides

top horizontal side: $y=1$
bottom: $y=-1$

Fix y, say $y=-\frac{1}{2}$

Notice

$$\frac{\sqrt{3}}{2} = \sqrt{1-(-\tfrac{1}{2})^2}$$

For each y btwn -1 & 1 x goes from $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$

so

$$\iint_R f(x,y)\,dx\,dy = \int_{y=-1}^{y=1}\int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

↳ This used ②

↳ can also use ③. Then your vertical lines are $x=-1$ and $x=1$ then you get:

$$\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} f\,dy\,dx$$

e.g. $f(x,y)=1$ (const fcn)

Recall

$$\iint_R 1\,dx\,dy = \text{area}(R)$$

Try this for R = unit disc.

$$\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} 1\,dy\,dx$$

$$= \int_{x=-1}^{x=1} y\Big]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}$$

$$= \int_{x=-1}^{x=1} (\sqrt{1-x^2}) - (-\sqrt{1-x^2})\,dx$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

↖ This is the integral we know from single-var calc for the area of a circle.

↖ can use Trig Sub to evaluate

Trigonometric Substitution uses change of variables formula in calculus can use Trig Sub to evaluate

Trignometric Substitution uses change of variables formula in calculus.

Review

$x = \cos u$

$$\int_{x=-1}^{x=1} 2\sqrt{1-x^2}\,dx = \int_{x=-1}^{x=1} 2\,|\sin u|\,dx$$

Change of limits of integration

$x=-1 \quad \cos(u) = -1 \implies u = \pi$

$x = 1 \quad \cos(u) = 1 \implies u = 2\pi$

$$\int_{u=\pi}^{u=2\pi} 2|\sin u|\,dx \longleftarrow$$

Note

- for $\pi \le u \le 2\pi$

  $\sin u \le 0$
- therefore

  $|\sin u| = -\sin u$

$$= \int_{u=\pi}^{u=2\pi} -2\sin u\,dx$$

$$= \int_{\pi}^{2\pi} 2\sin^2 u\,du$$

Key: fund thm of calc req taking antiderivative wrt variable inside the d.

need $dx \to du$

How?

$$dx = \frac{dx}{du}\cdot du$$

$$= -\sin u\,du$$

Key idea

$$dx = \frac{dx}{du}\,du$$

Q/ Can we find $\iint_R dx\,dy$ using polar coords? (w/ R = unit disc.)

Q/ why is this helpful?

A/ Unit disc R has simple description in polar coords (r,θ):

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

this is really just the rectangle $[0,1]\times[0,2\pi]$ in (r,θ) coords

↳ so this reduces to 1

$$\iint 1\,dx\,dy = \iint 1\,dx\,dy$$

R $\quad (r,\theta) \in [0,1]\times[0,2\pi]$

Q/ How to convert btwn $dx\,dy$ & $dr\,d\theta$?

ie $dx\,dy = [\text{what}]\,dr\,d\theta$

A/ say we want to diff (x,y) wrt (r,θ)

↳ that's what the [what should be]

Q/ What is x, y in terms of r, θ?

A/ $x = r\cos\theta$

$y = r\sin\theta$

↳ this is really a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

↳ its deriv. is a 2×2 matrix

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

attempted ans:

$$\frac{dx}{dy} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} dr\,d\theta$$

$$\Rightarrow \iint dx\,dy = \iint_{[0,1]\times[0,2\pi]} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} dr\,d\theta$$

⚠ PROBLEM ⚠

↳ this is a matrix, not a scalar & A/V are scalar

Q/ How to turn a matrix into a scalar?

A/ determinant!

$$\frac{dx}{dy} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right) dr\,d\theta$$

In this case:

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

$$\Rightarrow \iint 1\,dx\,dy = \int^{\theta=2\pi}\int^{r=1} r\,dr\,d\theta$$

$$\Rightarrow \iint_R 1\,dx\,dy = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r\,dr\,d\theta$$

Using method ①

$$\int_{\theta=0}^{\theta=2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = 2\pi\left(\frac{1}{2}\right) = \pi$$

Q/ What does $\iint_R f(x,y)\,dx\,dy$ mean?

Recall

$\int_{[a,b]} f(x)dx$ is essentially just the value of f times the length (or change in x) of the interval.

Caveat f doesn't necessarily take one single value of the whole interval

Solution Riemanns sums!

↳ break [a,b] into little pieces on which f doesn't vary too much

↳ so, you can think of f as having approx constant value on each interval

↳ then make more & more intervals, w/ smaller & smaller intervals and take the limit as the mesh goes to 0

↳ max len of an interval among the intervals you broke [a,b] into

- Usually just use intervals that each have length $\frac{b-a}{N}$ $\Rightarrow$ mesh = $\frac{b-a}{n}$

↳ now say

Triple Integrals

Tuesday, March 16, 2021 12:22 PM

2. We talked about how to calculate double integrals. Now: theory

This time...
- use Riemann sum to explain change of vars
- introduce triple-integrals

Usual Riemann Sum:
Idea 0

Problem: $f(x)$ varies as x goes from a to b
Solution: Break up $[a, b]$ into little pieces on which f is $\approx$ a const
(bc f can't change too much on a small enough interval)
— precisely: continuity
- usual way to break up $[a,b]$, is into N intervals, each of the same length
- $I_i = \left[a + \frac{(i-1)(b-a)}{N}, a + \frac{i(b-a)}{N} \right]$

I is partitioned into the I_i
↳ Def: A partition of I is a way of breaking I into smaller intervals
$I = I_1 \cup I_2 \cup I_3 \cup \dots \cup I_n$
so that the interiors of smaller intervals don't overlap

Note: If $I_i = [a_i, b_i]$, its interval is (a_i, b_i)
- eg $[0,1] = [0, \frac{1}{3}] \cup [\frac{1}{3}, \frac{1}{2}] \cup [\frac{1}{2}, \frac{5}{8}] \cup [\frac{5}{8}, 1]$
- mesh of a partition is max length (I_i)

eg mesh $\frac{3}{8}$

For a region R in $\mathbb{R}$, a partition of R is a decomposition $R = R_1 \cup R_2 \cup \dots \cup R_N$
mesh(partition) = max area (R_i)

eg $R = [a,b] \times [c,d]$
choose M, and have $N = M^2$ little rectangles indexed by $i, j = 1, \dots, M$

$$\left[a + \frac{(i-1)(b-a)}{N}, a + \frac{i(b-a)}{N} \right] \times \left[c + \frac{(j-1)(d-c)}{M}, c + \frac{j(d-c)}{M} \right]$$

area of $R_{ij} = \frac{(b-a)(d-c)}{M} =$ mesh

Back to 1-D:
Given a partition $I = I_1 \cup I_2 \cup \dots \cup I_N$

Back to 1-D:

Given a partition $I = I_1 \cup I_2 \cup \dots \cup I_N$

Choose $x_i \in I_i \ \forall i$

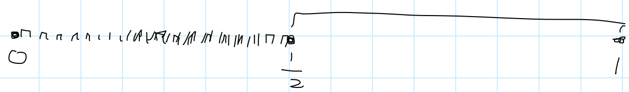
$$\text{Riemann sum} = \sum_{i=1}^{N} f(x_i) \cdot \text{length}(I_i)$$

$$\int_a^b f(x)\,dx = \int_I f(x)\,dx = \lim_{\text{mesh}\to 0} \sum_{i=1}^{N} f(x_i)$$

- mesh being small $\Rightarrow$ every subinterval is "little enough"

Q/ Why can't just take $N \to \infty$?

What if $I = [0,1]$



So divide $[0, \frac{1}{2}]$ into $N-1$ pieces and take $I_N = [\frac{1}{2}, 1]$

then as $N \to \infty$, the # of subintervals $\to \infty$

but the mesh stays $\frac{1}{2}$

$\Rightarrow$ need mesh to approach 0

Precisely $\lim_{\text{mesh}\to 0} \sum_{i=1}^{N} f(x_i)\cdot \text{length}(I_i) = L$

means $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for any partition $I = I_1 \cup \dots \cup I_N$ of mesh $< \delta$ and any choice of $x_i \in I_i \forall i$

$$\left| L - \sum_{i=1}^{N} f(x_i) \,\text{len}(I_i) \right| < \varepsilon$$

Thm

$\lim_{\text{mesh}\to 0} \sum_{i=1}^{N} \dots$ exists for f continuous from a to b and is the integral as we know it

---

2 Dimensions:

Recall a partition of R is a decomp $R = R_1 \cup R_2 \cup \dots \cup R_N$ whose interiors don't overlap.

$$\iint_R f(x,y)\,dx\,dy = \lim_{\text{mesh}\to 0} \sum_{i=1}^{N} f(x_i, y_i)\,\text{area}(R_i)$$

of the partition

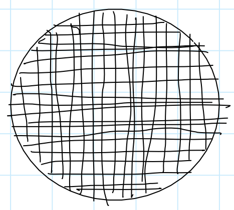
$(x_i, y_i) \in R_i$

eg Divide $[a,b] \times [c,d]$ into rectangles as above.

eg Divide $[a,b] \times [c,d]$ into rectangles as above.

eg Sierpinski's triangle

eg a circle



→ technically need diameters to approach 0

Note: theory of Riemann sums and mesh is theory — use it to prove general facts about integration but don't compute w/ it directly

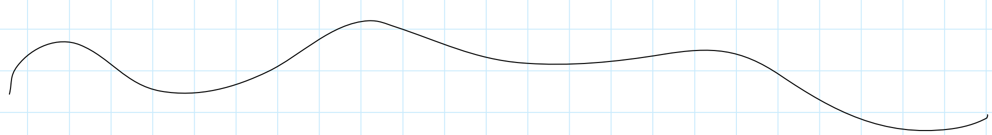
e.g.

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dx\,dy$$ FUBINI'S THEOREM

$$\iint\limits_{R_1\cup R_2} f(x,y)\,dx\,dy = \iint\limits_{R_1} f(x,y)\,dx\,dy + \iint\limits_{R_2} f(x,y)\,dx\,dy$$

if R_1 & R_2 have disjoint interiors

Proof let $L = \iint\limits_{R_1\cup R_2}$ $L_1 = \iint\limits_{R_1}$ $L_2 = \iint\limits_{R_2}$



Now the Riemann sum over $R_1 \cup R_2$ is the sum of the Riemann sums over each individual region R_1 and R_2

$$\Rightarrow \left| \iint\limits_{R_1\cup R_2} f\,dx\,dy - \iint\limits_{R_1} f\,dx\,dy - \iint\limits_{R_2} f\,dx\,dy \right| < \varepsilon$$

$$\iint\limits_{R_1\cup R_2} f\,dx\,dy - \iint\limits_{R_1} f\,dx\,dy - \iint\limits_{R_2} f\,dx\,dy = 0$$

■

Triple Integration

Suppose $f: D \to \mathbb{R}$ for $D \subseteq \mathbb{R}^3$ (open) and $R \subseteq D$

Rough idea

$$\iiint_R f\,dx\,dy\,dz = f(x,y,z)\cdot \text{volume}(R)$$

A partition of $R = R_1 \cup \dots \cup R_N$ has a mesh

$$\iiint_R f\,dx\,dy\,dz = \lim_{\text{mesh}\to 0} \sum_{i=1}^{N} f(x_i, y_i, z_i)\,.\text{volume}(R_i)$$

$(x_i, y_i, z_i) \in R$

↳ calculate in a similar way as in 2-D

eg

$R = [a,b]\times[c,d]\times[e,f]$

$$\iiint_R g(x,y,z)\,dx\,dy\,dz$$

$$= \int_e^f\int_c^d\int_a^b g(x,y,z)\,dx\,dy\,dz$$

$R = \text{unit ball} = \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, x^2+y^2+z^2\le 1\right\}$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} f\,dz\,dy$$

↳ easier in spherical coords

Btwn cylindrical & cartesian

Recall

$dx\,dy = r\,dr\,d\theta$

$\Rightarrow dx\,dy\,dz = (dx\,dy)\,dz = (r\,dr\,d\theta)\,dz$

$= r\,dr\,d\theta\,dz$

Center of mass

Suppose we have n objects indexed by $i = 1,\dots,n$

where the ith object is at location

$\vec{r}_i = (x_i, y_i, z_i)$

and has mass m_i

Then center of mass is

vector sum

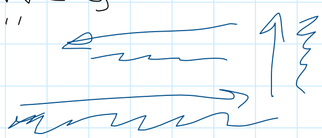
n

and has mass m_i

Then center of mass is vector sum

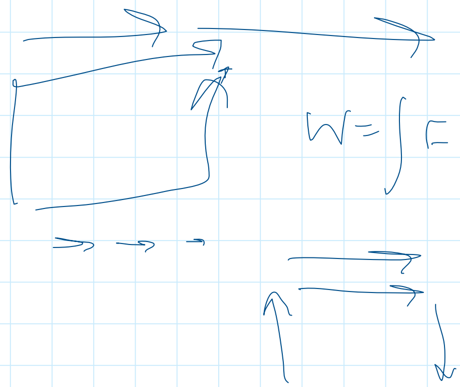
$$\frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} =$$

"weighted average of the locations of the objects — weighted by mass"



vector sum means: x-coord of center of mass is

$$\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$



$W = \int F \cdot dr$

y coord:

$$\frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

z-coord:

$$\frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

These formulas assume each obj has all its mass in a single point / location

realistic: mass density: $\rho(x, y, z)$ in units of mass / volume

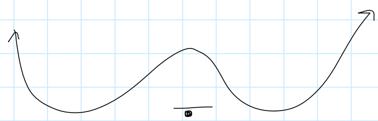
$$\text{center of mass} = \frac{\iiint \rho(x,y,z)\,\vec{r}\;dx\,dy\,dz}{\text{total mass} = \iiint \rho(x,y,z)\,dx\,dy\,dz}$$

Q/ What does $\iiint \rho(x, y, z)$ mean?
A/ output is

Integrals cont

Saturday, March 20, 2021 11:35 AM

skip 3.4, 3.7 in [Co]
responsible for: 3.1, 3.2, 3.3, 3.5, 3.6 in [Co]



Lagrange multiplier is solving for $\frac{d}{dt} = 0$

Suppose we integrate $\iiint_R f(x,y)\,dx\,dy$ where R is a region $\mathbb{R}^2$, f defined on R.

A partition P of R is a decomposition
$R = R_1 \cup R_2 \cup \cdots \cup R_N$
whose interiors don't intersect.

mesh (P) = max diameter (R_i)
→ Diameter$(R_i) = \sup_{\vec{v}_1, \vec{v}_2 \in R_i} \|\vec{v}_1 - \vec{v}_2\|$

diameter of a rectangle = length of diagonal
" " " $\triangle$ = " " longest side

Q: why diameter, not area, for mesh?
A. Consider $N \times \frac{1}{N^2}$ rectangle
↳ Area really small $\left(\frac{1}{N}\right)$
↳ Diameter: large
$= \sqrt{N^2 + \frac{1}{N^4}} \simeq N$

We want to avoid such an R_i
↳ want each R_i to be small in all directions

Now

$$\iint_R f(x,y)\,dx\,dy = \lim_{\text{mesh} \to 0} \sum_{i=1}^{N} f(x_i, y_i)\,\text{area}(R_i)$$
$(x_i, y_i) \in R_i$

Single integrals
replace area w/ length

triple integrals
replace area w/ volume

call area(R_i) "=" $\Delta A = \Delta$ area
↳ for triple integrals, $\Delta V = \Delta$volume

Q/ What is $dx\,dy$?
A/ it's like $\Delta x \Delta y$ = area of Δy $\boxed{}$ Δx

= Δ area

eg if $R = [a,b] \times [c,d]$, divide into $N = M^2$ little rectangles, all congruent to each other.
Then each rectangle is Δx by Δy
$\Delta x = \frac{b-a}{M}$ $\Delta y = \frac{d-c}{M}$

↳ in 3D

$$\Delta x = \frac{b-a}{M} \qquad \Delta y = \frac{d-c}{M}$$

↳ in 3D

$$\Delta V = \Delta x \, \Delta y \, \Delta z$$

Let's explain change of variables formula using this idea

Suppose we have coords u,v and x,y and a coordinate transformation

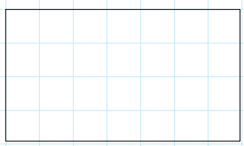
$$x = g_1(u,v) \qquad g = (g_1, g_2)$$

$$y = g_2(u,v) \qquad (x,y) = g(u,v)$$

Q/ How to relate $dx\,dy$ to $du\,dv$?

A/ suppose we have a little rectangle in coords u,v like so:

$C = (u_i, v_i + \Delta v)$ $\quad$ $(u_i + \Delta u, v_i + \Delta v) = D$



$A = (u_i, v_i)$ $\quad$ $(u_i + \Delta u, v_i) = B$

area of this rectangle (in u,v coords) is $\Delta u \Delta v$ "=" $du\,dv$

Q/ If we apply g to this rectangle, what (approximately) is the area is xy-coords of the resulting shape?

approximation should better as $\Delta u, \Delta v \to 0$

set $(x_i, y_i) = g(u_i, v_i) = g(A)$

$$g(B) = g(u_i + \Delta u, v_i)$$

gets better as $\Delta u \to 0$

$$\approx g(u_i, v_i) + \Delta u \cdot \frac{\partial g}{\partial u}(u_i, v_i)$$

$$= (x_i, y_i) + \Delta u \left(\frac{\partial g_1}{\partial u}(u_i, v_i), \frac{\partial g_2}{\partial u}(u_i, v_i) \right)$$

$$g(C) = g(u_i, v_i + \Delta v)$$

$$\approx g(u_i, v_i) + \Delta v \frac{\partial g}{\partial v}(u_i, v_i)$$

$$= (x_i, y_i) + \Delta v \left(\frac{\partial g_1}{\partial v}(u_i, v_i), \frac{\partial g_2}{\partial v}(u_i, v_i) \right)$$

$$g(D) = g(u_i + \Delta u, v_i + \Delta v)$$

$$= (x_i, y_i) + \Delta u \frac{\partial g}{\partial u}(u_i, v_i) + \Delta v \frac{\partial g}{\partial v}(u_i, v_i)$$

→ So, g applied to the rectangle has verticies

$$g(A), g(B), g(C), g(D)$$

$$\approx (x_i, y_i), (x_i, y_i) + \vec{r}_1, (x_i, y_i) + \vec{r}_2, (x_i, y_i) + \vec{r}_1 + \vec{r}_2$$

where

$$\vec{r}_1 = \Delta u \frac{\partial g}{\partial u}(u_i, v_i) = \Delta u \left(\frac{\partial g_1}{\partial u}(u_i, v_i), \frac{\partial g_2}{\partial u}(u_i, v_i) \right)$$

$$\vec{r}_2 = \Delta v \frac{\partial g}{\partial v}(u_i, v_i) = \Delta v \left(\frac{\partial g_1}{\partial v}(u_i, v_i), \frac{\partial g_2}{\partial v}(u_i, v_i) \right)$$

Note by Thm 1.B in [Co], the area of this parallelogram is $\|\vec{r}_1 \times \vec{r}_2\|$

$$= \left\| \left(0, 0, \Delta u \frac{\partial g_1}{\partial u} \Delta v \frac{\partial g_2}{\partial v} - \Delta u \frac{\partial g_2}{\partial v} \Delta v \frac{\partial g_1}{\partial v}\right) \right\|$$

$$= \left| \Delta \frac{\partial g_1}{\partial u} \Delta \frac{\partial g_2}{\partial v} - \Delta u \frac{\partial g_2}{\partial u} \Delta v \frac{\partial g_1}{\partial v} \right|$$

$$= \Delta u \Delta v \left| \frac{\partial g_1}{\partial u} \frac{\partial g_2}{\partial v} - \frac{\partial g_2}{\partial u} \frac{\partial g_1}{\partial v} \right|$$

this is the area in the xy plane

$$d\left(\substack{\text{area in}\\ \text{xy plane}}\right) = dx\,dy$$

$$= du\,dv \left| \frac{\partial g_1}{\partial u} \frac{\partial g_2}{\partial v} - \frac{\partial g_2}{\partial u} \frac{\partial g_1}{\partial v} \right|$$

this is the change of variables Formula

Note $dx\,dy$ really means $d(\text{Area})$ where area is taken in xy coordinates

like choosing a unit of measure for area (like m^2 vs ft^2)

This formula relates area in xy-coords in uv-coords

Q/ What about 3 dim?

A/ Use a 3×3 determinant = Volume of a parallelpiped

To compute w/ change of variables in 3-dim, use formula in book.

Below here is NOT IN SCOPE

Remark determinants in general are the scaling factor for n-volume

1-volume = length

2-volume = area

3-volume = volume

4-volume = hypervolume

Remark What if we want to use the determinant instead of its absolute value?

signed area vs area

signed area = $\pm$ area

and it's $(-)$ if opposite orientation

Note:

If using signed area, must keep track of the order of x and y.

For us:

$$\iint f\,dx\,dy = \iint f\,dy\,dx$$

For signed area

$$\iint f\,dx \wedge dy \quad \text{and} \quad dy \wedge dx = -dx \wedge dy$$

Why is it useful?

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$\Rightarrow dx \wedge dy =$$

$$\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)\wedge\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\right)$$

$$=\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}du\wedge dv+\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}dv\wedge du+(\text{stuff})\,du\wedge du+(\text{stuff})\,dv\wedge dv$$

Note

1. $du\wedge du=-du\wedge du$
   $\Rightarrow du\wedge du=dv\wedge dv=0$
2. $dv\wedge du=-du\wedge dv$

Now no absolute value, these $\wedge$ have to do with exterior powers & differential forms

Line integrals

Tuesday, March 30, 2021 11:12 AM

In single-var, derivatives and integrals are essentially opposites

$$\frac{df}{dx} = g$$

FTC: $\int_a^b g(x)\,dx = f(b) - f(a)$

So far we have the following in multivar
① Differentiation → partial differentiation
Given $f(x,y)$ (two inputs, one output)
have $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ or together:

gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ (two outputs? inputs)

Q/ Given ∇f (and maybe some initial cond $f(x_0, y_0) = z_0$ can we integrate?

We learned about a kind of integration in multivariable: takes a function $g(x,y)$ with 2 inputs and one output and a region R.
Then

$$\iint_R g(x,y)\,dx\,dy \text{ is a number}$$

(one output)

Want something like FTC:
ie, given $\vec{a} \in \mathbb{R}^2$ and $\vec{b} \in \mathbb{R}^2$
<u>want</u>

$$f(\vec{b}) - f(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \nabla f\, d?$$

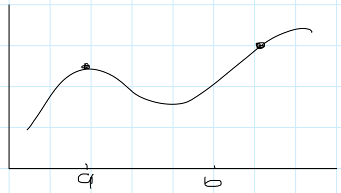
if we write the RHS in terms of the double integration we learned, get 2 problems:
① ∇f has 2 outputs, not 1
② How to choose R in terms of $\vec{a}$ and $\vec{b}$?

Need a new kind of integration s.t.

$$f(\vec{b}) - f(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \nabla f\, d(\text{something})$$

this will be line integration [Co]138

Recall what happens in 1D



Q/ How to find $f(b)-f(a)$ in terms of $f'(x)$?

A/ <u>Idea</u> divide $[a,b]$ into little pieces $I_1, \dots, I_N$ eg $I_1 = \left[a, a+\frac{(b-a)}{N}\right]$

$$x_i = a + \frac{i(b-a)}{N} \quad \text{so} \quad I_i = [x_{i-1}, x_i]$$

$$\Delta x_i = \text{length}(I_i) = \frac{b-a}{N} = x_i - x_{i-1}$$

Q/ What is Δf over I_i?

A/ $f(x_i) - f(x_{i-1})$ ← exact

$\Delta x_i \cdot f'(x_i)$ ← approx

↳ gets better as $N \to \infty$

$$f(b) - f(a) = f(x_N) - f(x_0) = \sum_i \Delta f$$

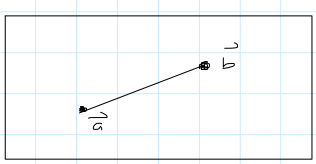
$$= \sum_i f(x_i) - f(x_{i-1})$$

$$= f(x_1) - f(x_0) + f(x_2) - f(x_1) + \dots + f(x_n) - f(x_{N-1})$$

$$\approx \sum_i \Delta x_i \cdot f'(x_i)$$

← Riemann sum whose $\lim_{N \to \infty}$ is $\int_a^b f'(x)\,dx$

<u>Now, in 2D</u>



$\mathbb{R}^2$

$$(x_i, y_i) = \vec{a} + \frac{i(\vec{b} - \vec{a})}{N}$$

<u>Notice</u>

$(x_0, y_0) = \vec{a}$ $\quad (x_N, y_N) = \vec{b}$

<u>Now</u>

$$f(\vec{b}) - f(\vec{a}) = f(x_N, y_N) - f(x_0, y_0)$$

$$= \sum_{i=1}^{N} f(x_i, y_i) - f(x_i, y_{i-1})$$

$$= \sum^{N} \Delta f$$

$$\sum_{i=1} f(x_i, y_i) - f(x_i, y_{i-1})$$

$$= \sum_{i=1}^{N} \underbrace{\Delta F}_{(x_{i-1},\, y_{i-1}) \to (x_i,\, y_i)}$$

Q/ How to estimate ΔF using derivatives?

A/

<u>Notice</u> we have
$$\Delta x_i = x_i - x_{i-1}$$
$$\Delta y_i = y_i - y_{i-1}$$

$$\Delta F \simeq \Delta x_i \frac{\partial F}{\partial x}(x_i, y_i) + \Delta y_i \frac{\partial F}{\partial y}(x_i, y_i)$$

So
$$F(\vec{b}) - F(\vec{a}) = \sum_{i=1}^{N} \Delta F$$

$$\simeq \sum_{i=1}^{N} \Delta x_i \frac{\partial F}{\partial x} + \Delta y_i \frac{\partial F}{\partial y}$$

will define line integrals using lines like this

$$= \sum_{i=1}^{N} (\Delta x_i, \Delta y_i) \cdot \underbrace{\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial x}\right)}_{\nabla F}$$

$$= \sum_{i=1}^{N} \nabla F \cdot (\Delta x_i, \Delta y_i)$$

$$= \sum_{i=1}^{N} \nabla F \cdot \Delta \vec{r_i}$$

We call
$$\lim_{N \to \infty} \sum_{i=1}^{\infty} \nabla F \cdot \Delta \vec{r_i} = \int_{\vec{a}}^{\vec{b}} \nabla F \cdot d\vec{r}$$

$$= \int_{\vec{a}}^{\vec{b}} \nabla F \cdot (dx, dy)$$

$$= \int_{\vec{a}}^{\vec{b}} \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) \cdot (dx, dy)$$

$$= \int_{\vec{a}}^{\vec{b}} \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

In general, for a fcn w/ 2 outputs & inputs (like ∇F)
we can define

$$(P(x,y), Q(x,y)) = (P, Q)$$

we can define
$$\oint_{\vec{a}}^{\vec{b}} (P, Q) \cdot d\vec{r} = \int_{\vec{a}}^{\vec{b}} P\,dx + Q\,dy$$

$$= \lim_{N \to \infty} \sum_{i=1}^{N} P(x_i, y_i) \cdot \Delta x_i + Q(x_i, y_i) \cdot \Delta y_i$$

When we defined $\int_{\vec{a}}^{\vec{b}} (P, Q) \cdot (dx, dy)$, we set

$$(x_i, y_i) = \vec{a} + \frac{i(\vec{b} - \vec{a})}{N}$$

such a point is on the line segment from $\vec{a}$ to $\vec{b}$.

More generally, we can integrate $(P, Q) \cdot (dx, dy)$ along any $\underbrace{\text{curve}}$ from $\vec{a}$ to $\vec{b}$.

↳ 1D subset of $\mathbb{R}^2$ that has a connected start and endpoint

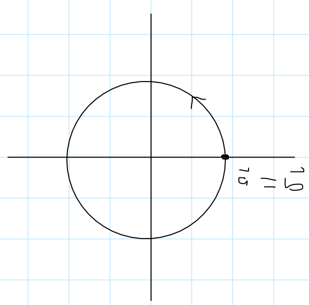
If $\vec{a} = \vec{b}$, we call a curve from $\vec{a}$ to $\vec{b}$ a loop

→ trivial loop stays at $\vec{a}$

Often, we parameterize a path, using t in some interval in $\mathbb{R}^1$

eg $\vec{a} = \vec{b} = (1, 0)$ in $\mathbb{R}^2$

consider the loop given by the unit circle (counterclockwise)



Q/ How to parameterize?
(You can parameterize same path in diff ways.

eg

$(x(t), y(t)) = (\cos(t), \sin(t))$
$t \in [0, 2\pi]$

$(x(t), y(t)) = (\cos(2\pi t), \sin(2\pi t))$
$t \in [0, 1]$

$(x, y) = (\cos(2\pi t^2), \sin(2\pi t^2))$
$t \in [0, 1]$

A/ Given a path C from $\vec{a}$ to $\vec{b}$, will define

$$\int_C (P, Q) \cdot (dx, dy)$$

To calculate it, we need to choose a parameterization of C, but the value of the integral is ind to the parameterization

Abstract definition of $\int_C P\,dx + Q\,dy$

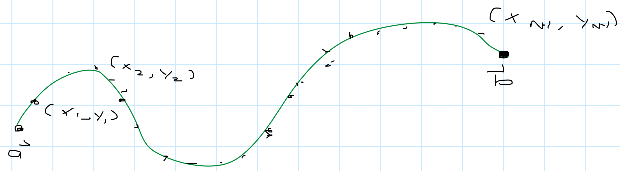
For each N, choose a partition of C into smaller paths.

Suppose C is from $\vec{a}$ to $\vec{b}$.

↳ set $(x_0, y_0) = \vec{a}$

$(x_N, y_N) = \vec{b}$

↳ Choose (x_1, y_1), (x_2, y_2), generally (x_i, y_i) on path C.



form Riemann sum:

$$\sum_{i=1}^{N} P(x_i, y_i) \cdot \Delta x_i + Q(x_i, y_i) \cdot \Delta y_i)$$

$$\Delta x_i = x_i - x_{i-1}$$

$$\Delta y_i = y_i - y_{i-1}$$

want as $N \to \infty$, the $\max(\Delta x_i)$ and ____ go to 0

equivalently: mesh = $\max \Delta x_i, \Delta y_i$

$$\int_C P\,dx + Q\,dy = \lim_{N \to \infty} \sum_{i=1}^{N} P(x_i, y_i)\Delta x_i + Q(x_i, y_i)\Delta y_i$$

$$= \lim_{\text{mesh} \to 0} \sum \text{\_\_\_\_}$$

$$\int_C P\,dx + Q\,dy = \int_C P\,dt\frac{dx}{dt} + Q\,dt\,\frac{dx}{dt}$$

$$= \int_{t=a}^{t=b} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} \right) dt$$

Fundamental Theorem, Conservative Vector Fields

Thursday, April 1, 2021 11:14 AM

Suppose we have a curve/path C from $\vec{a}$ to $\vec{b}$.

$\vec{F}(x,y) = P(x,y)\,\hat{\imath}$
$\vec{r} = x\hat{\imath} + y\hat{\jmath} = (x,y)$
$(d\vec{r} = (dx, dy)$ etc$)$

Then we defined $\int_C P\,dx + Q\,dy = \int_C \vec{F}\cdot d\vec{r}$

$$= \lim_{\text{mesh}\to 0} \sum_{i=1}^{N} P(x_i, y_i)\,\Delta x_i + Q(x_i, y_i)\,\Delta y_i$$

$$= \lim \sum \vec{F}(x_i, y_i)\cdot \Delta\vec{r}_i$$

where $\Delta x_i = x_i - x_{i-1}$ $\quad \Delta y_i = y_i - y_{i-1}$
mesh $= \max \| (x_i, y_i) - (x_{i-1}, y_{i-1}) \|$
(recall $\|\vec{v} - \vec{w}\|$ = distance from $\vec{v}$ to $\vec{w}$)
where
$\vec{a} = (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots, (x_{N-1}, y_{N-1}), (x_N, y_N) = \vec{b}$
is a sequence of points on C.

" $\lim_{\text{mesh}\to 0}$ " means the limit over all such sequences as the mesh approaches 0.

i.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. the Riemann sum is w/in ε of the integral for any such sequence with mesh $< \delta$

Warning

① If we go from $\vec{b}$ to $\vec{a}$ along C (in the opposite direction), we call this path $-C$.
then $\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$

Why? if $(x_0, y_0), (x_1, y_1), \ldots, (x_{N-1}, y_{N-1}), (x_N, y_N)$ is along C
then
$(x_N, y_N), (y_{N-1}, y_{N-1}), \ldots, (x_1, y_1), (x_0, y_0)$ goes along $-C$
so
you negate the Δx_i.

② If C is a loop (closed loop) like $\vec{a} = \vec{b}$, then you must specify the direction of C. And if you reverse the direction, you get negative of the circle.
eg if C is a circle

Compare w/ single variable

$$\int_0^1 x^2\,dx = \frac{1}{3} \qquad \int_1^0 x^2\,dx = -\frac{1}{3}$$

$$\int_a^b f'(x)\,dx = f(b) - f(a) \text{ is true even if } a > b.$$

How to compute?

Choose a parameterization, ie a pair of fcns $x(t), y(t)$ defined for $t \in [a,b]$
s.t. $\vec{a} = (x(a), y(a))$
$\vec{b} = (x(b), y(b))$
and $(x(t), y(t))$ goes along the path C as t goes from a to b.

Now $dx = \frac{dx}{dt}\,dt \qquad dy = \frac{dy}{dt}\,dt$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \int P\frac{dx}{dt}\,dt + Q\frac{dy}{dt}\,dt$$

$$= \int_C (P\,x'(t) + Q\,y'(t))\,dt$$

Note $\int_C = \int_{t=a}^{t=b}$

eg $P(x,y) = x^2 - y^2$
$Q(x,y) = 3x - e^y$

Consider a segment C of a parabola:

eg $P(x,y) = x^2 - y^2$
$Q(x,y) = 3x - e^y$

Consider a segment C of a parabola:
given by
$(x,y) = (t, t^2)$ from $t=1$ to $t=2$

then $\int_C P\,dx + Q\,dy =$

$$= \int_{t=1}^{t=2} (P(x(t), y(t))\, x'(t) + (Q(x(t), y(t))\, y'(t)$$

$$= \int_1^2 \left((t^2 - t^4) + (3t - e^{t^2})(2t) \right) dt$$

$$= \int_1^2 t^2 - t^4 + 6t^2 - 2te^{t^2}\, dt$$

$$= \left[\frac{7t^3}{3} - \frac{t^5}{5} - e^{t^2} \right]_{t=1}^{t=2}$$

$$= \left(\frac{56}{3} - \frac{32}{5} - e^4 \right) - \left(\frac{7}{3} - \frac{1}{5} - e \right)$$

$$= \frac{49}{3} - \frac{31}{5} + e - e^4$$

Note og definition did not depend on a parameterization.
eg what if we used
$(x,y) = (\sqrt{t}, t)$ for $t \in [1,4]$
$\Rightarrow$ the fact that there's a definition independent of parameterization implies that we get the same result

What about 3 dim?

Exact same formula:

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{mesh \to 0} \sum_{i=1}^{N} \vec{F}(x_i, y_i, z_i) \cdot \Delta \vec{r}_i$$

now C is a path in $\mathbb{R}^3$
from $\vec{a} = (x_0, y_0, z_0)$
to $\vec{b} = (x_N, y_N, z_N)$
and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
$\Delta \vec{r}_i = (\Delta x_i, \Delta y_i, \Delta z_i)$
and mesh $= \max_i \| (x_i, y_i, z_i) - (x_{i-1}, y_{i-1}, z_{i-1}) \|$

Note $\vec{F}$ must have 3 components bc we take its dot product with $\Delta \vec{r}_i$ and now $\Delta \vec{r}_i$ has 3 components

so $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$

In n-dim : $\vec{F}$ should have n outputs but C is still a path (1-dim) in $\mathbb{R}^n$

Note if $(x(t), y(t))$ for $t \in [a,b]$ is a parameterization of C
then $x(a+b-t), y(a+b-t)$ for $t \in [a,b]$ is a parametriz. of $-C$
Notice:
$(x(a+b-a), y(a+b-a)) = (x(b), y(b))$
$(x(a+b-b), y(a+b-b)) = (x(a), y(a))$
Q/ How does this negate the integral?
A/ bc it negates $x'(t)$ and $y'(t)$
ie. $\frac{d(x(a+b-t))}{dt} = -\frac{dx}{dt}$

4.1 [Co] $\to$ diff kind of line integration
instead of $\int_C \vec{F} \cdot d\vec{r}$ we do

$\int_C F\,ds$ where s is arclength

$$ds = \|dr\| = \sqrt{dx^2 + dy^2}$$

In Riemann sum terms:

$$\int f\,ds = \lim_{m \to 0} \sum_{i=1}^{N} f(x_i, y_i)\,\Delta s_i$$

where $\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \| (x_i, y_i) - (x_{i-1}, y_{i-1}) \|$

<u>How to calc?</u>

$dx = \frac{dx}{dt}\,dt \quad dy = \frac{dy}{dt}\,dt$

so $ds = \sqrt{dx^2 + dy^2}$

<u>How to calculate?</u>

$dx = \frac{dx}{dt}\,dt \quad dy = \frac{dy}{dt}\,dt$

so $ds = \sqrt{dx^2+dy^2} = \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)dt^2}$

so $ds = \sqrt{dx^2+dy^2} = \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)dt^2}$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\Longrightarrow \int_C f\,ds = \int f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

<u>Note</u>

so $ds = \sqrt{dx^2+dy^2} = \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)dt^2}$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\Longrightarrow \int_C f\,ds = \int f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

<u>Note</u>

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt}\vec{T} \qquad \text{(Chap 1)}$$

$$\Longrightarrow d\vec{r} = \underbrace{ds}_{\text{scalar}}\,\underbrace{\vec{T}}_{\text{vector}}$$

$$\Longrightarrow \vec{F}\cdot d\vec{r} = \vec{F}\cdot\left(ds\,\vec{T}\right) = ds\left(\vec{F}\cdot\vec{T}\right)$$

$$\Longrightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\vec{T}\right) ds$$

$$\Longrightarrow \vec{F}\cdot d\vec{r} = \vec{F}\cdot\left(ds\,\vec{T}\right) = ds\left(\vec{F}\cdot\vec{T}\right)$$

$$\Longrightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\vec{T}\right) ds$$

$$\Longrightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\vec{T}\right) ds = \int_C f\,ds$$

where $f = \vec{F}\cdot\vec{T}$ (dot prod of vectors is a scalar)

Recall $\vec{F}\cdot\vec{T}$ is the "component" of $\vec{F}$ in the $\vec{T}$-direction

so, eg if $\vec{F}$ and $\vec{T}$ are in the same direction,
then $\vec{F}\cdot\vec{T} = \|\vec{F}\|$

if $\vec{F}$ and $\vec{T}$ are $\perp$, the $\vec{F}\cdot\vec{T} = 0$

Recall $\vec{T}$ is tangent to the path C.

In physics: work = Force × distance (simple)

More sophisticated:

Force is a vector

displacement is a vector and

W = (force) • (displacement)

Force is a vector
displacement is a vector and
$W = (\text{force}) \cdot (\text{displacement})$

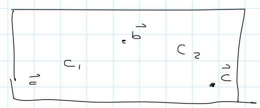
eg. If Force $\perp$ to the direction of motion (eg for an obj in circular orbit) then no work is done.

Line integrals $\int_C \vec{F} \cdot d\vec{r}$ allow us to calculate if $\vec{F}$ is the Force vector

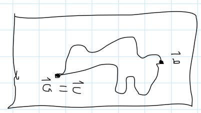
ie $\int_C \vec{F} \cdot d\vec{r}$ is the work done by the force done on an object as it goes along the path C.
(usually t = time)

If C_1 is a path from $\vec{a}$ to $\vec{b}$ and C_2 a path from $\vec{b}$ to $\vec{c}$, the $C_1 + C_2$ is the path from $\vec{a}$ to $\vec{c}$ given by going along C_1 then C_2.

eg



eg suppose $\vec{a} = \vec{c}$



then $C_1 + C_2$ is from $\vec{a}$ to $\vec{c}$
↳ closed path (loop)

Next time

$$\int_{C_1 + C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

Green's Theorem

Sunday, April 11, 2021 1:39 PM

A path/curve (called C or γ) starts at some $\vec{a}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and end at some $\vec{b}$.

→ If $\vec{a} = \vec{b}$ it's closed, so you must specify the direction.

→ Reversing the direction sends C to $-C$ and:

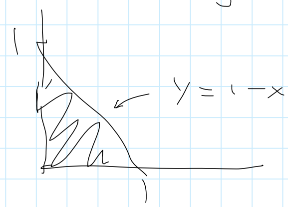
$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot dr$$

$$\int_{-C} F\,ds = \int_C F\,ds$$

$\boxed{1 - x - y = 0}$

$x + y + \cancel{z} = 1$

$$\left.\begin{matrix} x = 1 \\ y = 1 \end{matrix}\right\}$$



FTC for line integrals

Suppose C is a path/curve from $\vec{a}$ to $\vec{b}$, and F is a scalar function defined on an open domain $D \subseteq \mathbb{R}^2$ (or $\mathbb{R}^3$) containing C.

Let $\vec{F} = \nabla F$.

Then

$$\int_C \vec{F} \cdot d\vec{r} = F(\vec{b}) - F(\vec{a})$$

Remarks

① $\int_{-C} = -\int_C$ makes sense in terms of the FTC

bc if C is from $\vec{a}$ to $\vec{b}$, then $-C$ is from $\vec{b}$ to $\vec{a}$ and $F(\vec{a}) - F(\vec{b}) = -(F(\vec{b}) - F(\vec{a}))$

② FTC says $\int_C \vec{F} \cdot d\vec{r}$ only depends on the endpoints of C & not on the particular path btwn them if $\vec{F} = \nabla F$.

BUT for many $\vec{F}$, the integral does depend on the path

⇒ $\vec{F}$ is not a gradient of some F.

(converse is "basically" true)

Defn: If $\vec{F} = \nabla F$, Then F is called a potential for

Conservative ⇒ $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints

Defn: $\vec{F}$ is conservative if $\vec{F}$ has a potential.

③ If $\vec{F}$ is conservative, then $\int_C \vec{F} \cdot d\vec{r} = 0$ if C is closed.

(bc $F(\vec{a}) - F(\vec{a}) = 0$)

Recall If $(x(t), y(t))$ is a parameterization of a curve C for $a \le t \le b$, then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

$$= \int_a^b [P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)]\,dt$$

* similar for $\mathbb{R}^3$ but w/ z also.

independent of parameterization

Proof of FTC (in $\mathbb{R}^2$)

suppose $\vec{F} = \nabla F$ and choose a parameterization $(x(t), y(t))$ for $a \le t \le b$.

Proof of FTC (in $\mathbb{R}^2$)

suppose $\vec{f} = \nabla F$ and choose a parameterization $(x(t), y(t))$ for $a \le t \le b$.

Let $\vec{a}, \vec{b}$ be the endpoints of C.

$\Rightarrow$ $\vec{a} = (x(a), y(a)) = g(a)$

$\vec{b} = (x(b), y(b)) = g(b)$

$$\int_C \nabla F \cdot d\vec{r} = \int_C \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy$$

$$= \int_{t=a}^{t=b} \underbrace{\left[\frac{\partial F}{\partial x}(x(t), y(t))x'(t) + \frac{\partial F}{\partial y}(x(t), y(t))y'(t)\right]}_{\text{use MV chain rule to rewrite}} dt$$

$g : [a,b] \to \mathbb{R}^2 \quad g(t) = (x(t), y(t))$

so the derivative is $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$

$F : \mathbb{R}^2 \to \mathbb{R}$

so the derivative is $\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix}$

so the derivative of $F \circ g : [a,b] \to \mathbb{R}$ is

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \frac{\partial F}{\partial x} x'(t) + \frac{\partial F}{\partial y} y'(t)$$

= the integrand we had above

$= \frac{d}{dt}(F \circ g)$

$$\Rightarrow \quad = \int_{t=a}^{t=b} \left[\frac{\partial F}{\partial x}(x(t), y(t))x'(t) + \frac{\partial F}{\partial y}(x(t), y(t))y'(t)\right] dt$$

$$= \int_a^b \frac{d}{dt}(F \circ g)\, dt$$

$\underset{\text{by SV FTC}}{=} \left[F \circ g\right]_a^b$

$= F(g(b)) - F(g(a))$

$= F(\vec{b}) - F(\vec{a})$

■ QED

Suppose C_1 is a curve from $\vec{a}$ to $\vec{b}$ and C_2 is from $\vec{b}$ to $\vec{c}$, then get $C_1 + C_2$ from $\vec{a}$ to $\vec{c}$

"go along C_1, then along C_2"

Key fact

$$\int_{C_1 + C_2} \vec{f} \cdot d\vec{r} = \int_{C_1} \vec{f} \cdot d\vec{r} + \int_{C_2} \vec{f} \cdot d\vec{r}$$

Compare: $\displaystyle\int_a^b f\,dx + \int_b^c f\,dy = \int_a^c f\,dx$

Suppose we're given parameterization g_1 of C_1 and g_2 of C_2, each defined from $0 \le t \le 1$.

Q/ How to write parameterization g of $C_1 + C_2$ s.t. g is defined for $0 \le t \le 1$?

Q/ Given $t \in [0,1]$, what is $g(t)$?

A/

$$g(t) = \begin{cases} g_1(2t) & 0 \le t \le \frac{1}{2} \\ g_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

At $t = \frac{1}{2}$, the def is consistent because we assumed that the endpoint of C (aka $g_1(1)$) is the initial point of C_2 (aka $g_2(0)$)

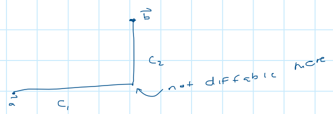
Technical point

We assumed that the endpoint of C (aka $g_1(1)$) is the initial point of C_2 (aka $g_2(0)$)

Technical point

When we compute line integrals using a param $(x(t), y(t))$ we take derivatives of the components. This requires that they are diff'able fcns of t.

<u>But</u> what if we have a curve C like this?



A/ We can still compute $\int_C$ by writing $C = C_1 + C_2$ and then $\int_C = \int_{C_1} + \int_{C_2}$ if C is piecewise smooth but not smooth

<u>Note</u> $C_1 + C_2 = C_1 \cup C_2$

Compatibility btwn adding curves and FTC:

For $\vec F = \nabla F$, the fact that

$$\int_{C_1 + C_2} = \int_{C_1} + \int_{C_2}$$

is equivalent to:

$$F(\vec c) - F(\vec a) = (F(\vec c) - F(\vec b)) + (F(\vec b) - F(\vec a))$$

<u>Recall</u>

if $\vec F$ is conservative then

② $\int_C \vec F \cdot d\vec r$ is path-independent

③ $\oint \vec F \cdot d\vec r = 0$ if C is closed

② $\iff$ ③ for any $\vec F$

<u>② $\Rightarrow$ ③</u> if C (from $\vec a$ to $\vec a$) is closed, let C' be the const path from $\vec a$ to $\vec a$

(ie C' parameterized by $g(t) = \vec a$)

then $g'(t) = 0$

$$\Rightarrow \int_{C'} \vec F \cdot d\vec r = \int \vec F(g(t)) \cdot g'(t)\, dt = \int 0\, dt = 0$$

<u>but</u> let C and C' have same endpoint

$$\Rightarrow \text{by ②} \int_C \vec F \cdot d\vec r = \int_{C'} \vec F \cdot d\vec r = 0$$

<u>③ $\Rightarrow$ ②</u> Suppose ③ is true and that C and C' have the same endpoints $\vec a$ and $\vec b$

<u>Consider</u> $-C'$ which is from $\vec b$ to $\vec a$ and

$C'' = C - C'$

$= C + (-C')$

= "go along C, then go along C' in the other direction"

$\Rightarrow C''$ is closed

by ③, $\int_{C''} \vec F \cdot d\vec r = 0$

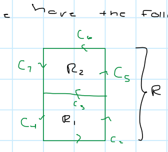
But $\int_{C''} \vec F \cdot d\vec r = \int_{C + (-C')} \vec F \cdot d\vec r$

$$= \int_C \vec F \cdot d\vec r + \int_{-C'} \vec F\, d\vec r = \int_C \vec F \cdot d\vec r - \int_{C'} \vec F\, d\vec r$$

$$\Rightarrow \int_C \vec F\, d\vec r = \int_{C'} \vec F \cdot d\vec r$$

■ QED Thm 43

Suppose we have the following curves.



$R_1 \to$ bottom □

$R_2 \to$ top □

$R \to$ whole □

Consider some vector field $\vec F = (P, Q)$

$$\int_{R_1} \vec{F}\cdot d\vec{r} = \int_{C_1+C_2+C_3+C_4} \vec{F}\cdot d\vec{r}$$

ccw
by default

$$\int_{R_2} \vec{F}\cdot d\vec{r} = \int_{C_5+C_6+C_7-C_3} \vec{F}\cdot d\vec{r}$$

$$\Rightarrow \int_{R_1} \vec{F}\cdot d\vec{r} + \int_{R_2} \vec{F}\cdot d\vec{r}$$

$$= \int_{C_1+C_2+\cancel{C_3}+C_4} \vec{F}\cdot d\vec{r} + \int_{C_5+C_6+C_7-\cancel{C_3}} \vec{F}\cdot d\vec{r} = \int_{R} \vec{F}\cdot d\vec{r}$$

$$= \int_{C_1+C_2+C_4+C_5+C_6+C_7} \vec{F}\cdot d\vec{r}$$

but going around ccw is $C_1+C_2+C_5+C_6+C_7+C_4$

More Green's Theorem

Saturday, April 17, 2021 9:01 AM

Recall

Def f is conservative if $f = \nabla F$ for some fcn F.

Technical point: If f is defined on a domain D (ie an open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$) we say "f is conservative on D" if there's a fcn F on D s.t. $f = \nabla D$

Will see an example of a vector field F:

(1) F is defined on D

(2) F is not conservative on D

(3) F is locally conservative on D

ie For each point, find a single field F that works for all of D.

Remark if D is connected, then any 2 potentials F_1 and F_2 for $\vec{f}$ must differ by a constant.

Definition $\vec{F}$ is path-independent (on D) if $\int_C \vec{f} \cdot d\vec{r}$ depends only on the endpoints of C (for $C \subseteq D$)

ie, if C_1, C_2 are curves in D w/ some endpoints (same direction), then

$$\int_{C_1} \vec{f} \cdot d\vec{r} = \int_{C_2} \vec{f} \cdot d\vec{r}$$

Thms (last time)

- conservative on D $\Rightarrow$ path independent on D
- $\vec{F}$ is path-ind. on D iff for every closed curve C in D, $\int_C \vec{f} \cdot d\vec{r} = 0$

Thm If $\vec{F}$ is path-ind on D, then $\vec{F}$ is conservative on D.

Proof sketch

Key if $\vec{F}$ path-ind on D, then for any two $P, Q \in D$, can define $\int_P^Q \vec{f} \cdot d\vec{r}$ w/o caring about which curve from P to Q we use.

Technical point true if D is connected. If not, thm still true but need to work seperately on each component of D.

Now choose $P_0 \in D$ and define

$$F(P) := \int_{P_0}^{P} \vec{f} \cdot d\vec{r}$$

Note: F must be path-ind for this to work

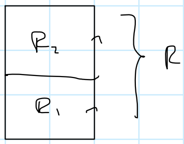
can check that $\nabla F = \vec{f}$

■ QED

Green's Theorem

→ see 4.3 [Co] ; notes linked

Last time



$$\int_R \vec{F} \cdot d\vec{r} = \int_{R_1} \vec{F} \cdot d\vec{r} + \int_{R_2} \vec{F} \cdot d\vec{r}$$

↖ default CCW

bc cancellation on the common edge intuitively, bc 2 gears going counterclockwise will grind against each other

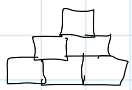
Generalization

If we have a grid:

R_{16}	17	18	19	20
11	12	13	14	15
6	7	8	9	10
R_1	R_2	R_3	R_4	R_5

then the integral $\vec{F} \cdot d\vec{r}$ around the whole perimeter is $\sum_{i=1}^{20} \int_{R_i} \vec{F}\, d\vec{r}$

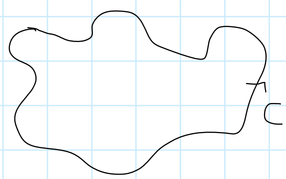
Note, the rectangles can be stacked in any way:



the integral around the outside perimeter is the sum of integrals along each of the little rectangles

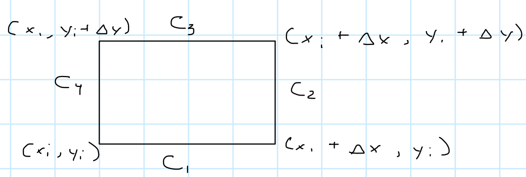
Idea:

Suppose we have a closed curve like so:



compute $\int_C \vec{F} \cdot d\vec{r}$ by breaking the region bounded by C into lots of little rectangles, and approximating $\vec{F}$ on each rectangle then adding it all up.

Now let's integrate around a tiny rectangle



say $\vec{F} = P\vec{i} + Q\vec{j}$

guess $\int_{C_1}$ cancels $\int_{C_3}$

ACTUALLY from C_1 to C_3, $\vec{F}$ changes by $\Delta y \cdot \frac{\partial \vec{F}}{\partial x}$

$$\frac{\partial \vec{F}}{\partial x} = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial Q}{\partial x}\vec{j}$$

idea $\vec{F}(x, y_i + \Delta y) \approx F(x_i, y_i) + \Delta y \frac{\partial \vec{F}}{\partial y}(x, y_i)$

let parameterize C_1 & C_3

Param of C_1:

$(x, y) = (x_i + t\Delta x, y_i)$ $\quad 0 \le t \le 1$

$(x'(t), y'(t)) = (\Delta x, 0)$

Param of C_3:

$(x, y) = (x_i + \Delta x - t\Delta x, y_i + \Delta y)$ $\quad 0 \le t \le 1$

$(x'(t), y'(t)) = (-\Delta x, 0)$

Now

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 (P(x_i + t\Delta x, y_i), Q(x_i + t\Delta x, y_i)) \cdot (\Delta x, 0)\, dt$$

$$= \int_0^1 (P(x_i + t\Delta x, y_i) \Delta x\, dt$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^1 F(x_i + \Delta x - t\Delta x, y_i + \Delta y) \cdot (-\Delta x, 0)\, dt$$

$$= -\int_0^1 P(x_i + \Delta x - t\Delta x, y_i + \Delta y) \Delta x\, dt$$

$$\approx -\int_0^1 \left[P(x_i + \Delta x - t\Delta x, y_i) + \Delta y \frac{\partial P}{\partial y}\right] \Delta x\, dt$$

$u = 1 - t$

$$= -\int_0^1 \left[P(x_i + u\Delta x, y_i) + \Delta y \frac{\partial P}{\partial y}\right] \Delta x\, du$$

$$= -\int_0^1 P(x_i + u\Delta x, y_i) \Delta x\, du - \int_0^1 \frac{\partial P}{\partial y} \Delta y \Delta x\, du$$

$$\int_{C_1 + C_3} \vec{F} \cdot d\vec{r} = \int_0^1 P(x_i + t\Delta x, y_i) \Delta x\, dt - \int_0^1 P(x_i + u\Delta x, y_i) \Delta x\, du - \int_0^1 \frac{\partial P}{\partial y} \Delta y \Delta x\, du$$

$$\approx -\int_0^1 \frac{\partial P}{\partial y} \Delta x \Delta y\, du$$

$$\approx -\frac{\partial P}{\partial y}(x_i, y_i) \Delta x \Delta y$$

Similarly

$$\int_{C_2 + C_4} \vec{F} \cdot d\vec{r} = \frac{\partial Q}{\partial x} \Delta x \Delta y$$

$$\Rightarrow \int_{\substack{\text{around} \\ \text{a little} \\ \text{rectangle} \\ \text{w/ sides} \\ \Delta x\ \&\ \Delta y}} \vec{F} \cdot d\vec{r} = \int_{C_1 + C_2 + C_3 + C_4} \vec{F}\, d\vec{r}$$

$\int$ around a little rectangle w/ sides Δx & Δy $\qquad \int_{C_1 + C_2 + C_3 + C_4}$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta x \Delta y$$

$$= \int_{\substack{\text{outer} \\ \text{perimeter}}} \vec{F} \cdot d\vec{r} = \sum_{\substack{\text{little} \\ \text{rectangles}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta x \Delta y$$

= Riemann sum for

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

where R is the region bounded by C

More Green's Theorem and Simple Connectedness

Saturday, April 17, 2021 9:01 AM

Recall in $\mathbb{R}^n$

Definition $\vec{F}$ is conservative in a region R if $\exists F$ a scalar fcn on R s.t $\vec{F} = \nabla F$, then "F is a potential for $\vec{F}$ on R"

Note If R is connected, then any 2 potentials for the same $\vec{F}$ differ by a constant.

Thm For $\vec{F}$ on R (connected):

conservative $\iff$ path independent $\iff$ $\int_C \vec{F} \, d\vec{r} = 0$ for C closed

Define for a vector field $\vec{F}$ in $\mathbb{R}^2$, the curl of $\vec{F}$ is the scalar fcn $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ for $\vec{F} = P\vec{i} + Q\vec{j}$

Proposition If $\vec{F}$ is conservative on R, then curl $\vec{F} = 0$

Proof Let F be a potential for $\vec{F}$ (on R)

Then

$$P = \vec{F} \cdot \vec{i} = \frac{\partial F}{\partial x}$$

$$Q = \vec{F} \cdot \vec{j} = \frac{\partial F}{\partial y}$$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

These are equal $\left(\frac{\partial}{\partial x \partial y} = \frac{\partial}{\partial y \partial x}\right)$

$$\Rightarrow \text{curl } \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

■ QED.

Next goal for certain regions R, curl $\vec{F} = 0 \Rightarrow \vec{F}$ is conservative

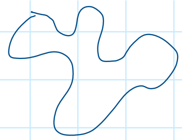
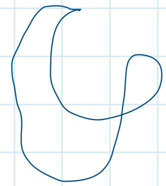
Jordan Curve Thm

Definition A simple closed curve C in $R \subseteq \mathbb{R}^2$ is a curve given by the parameterization $(x(t), y(t))$ for $a \leq t \leq b$ s.t.:

1. $(x(a), y(a)) = (x(b), y(b))$ ← closed
2. If $a \leq t_1 < t_2 \leq b$ then $(x(t_1), y(t_1)) \neq (x(t_2), y(t_2))$ unless $t_1 = a$? $t_2 = b$ ("doesn't cross itself")

e.g

Simple closed Not simple closed

Thm (Jordan Curve Thm)

If C is a simple closed curve in $\mathbb{R}^2$, then C divides $\mathbb{R}^2$ into 2 regions:

- one bounded by $\text{int}(C)$
- one unbounded $(\text{ext}(C))$

In fact, $\forall P \in \mathbb{R}^2$ either $P \in \text{ext}(C)$, $P \in C$ or $P \in \text{int}(C)$

eg $C = \left\{ \vec{r} \in \mathbb{R}^2 \;\middle|\; \| \vec{r} - \vec{r}_0 \| = r \right\}$

$= C_{r_0}(r)$

$=$ circle of radius r and center $\vec{r}_0$

(simple closed if $r > 0$)

$D = D_{\vec{r}_0}(r) = \left\{ \vec{r} \in \mathbb{R}^2 \;\middle|\; \| \vec{r} - \vec{r}_0 \| < r \right\}$

"open disc"

then $D = \text{int}(C)$

Remarks

1. C is a point iff $a = b$ (in parameterization) iff $\text{int}(C) = \emptyset$
2. $C = \text{boundary}(\text{int}(C)) =$ "$\partial(\text{int}(C))$"

Thm (Green's Thm)

Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ is defined & diff'able on some open domain $D \subseteq \mathbb{R}^2$

(technical point: need $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial y}$ all continuous)

and C is a simple closed curve in D s.t. $\text{int}(C) \subseteq D$

then
$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$
$$= \int_{\text{int}(C)} \text{curl}\, \vec{F} \, dx\,dy$$

Definition Let R be an open region in $\mathbb{R}^2$

We say R is simply connected if $\forall$ simple closed C contained in R, we also have $\text{int}(C) \subseteq R$

eg

- disc $D_{\vec{r}_0}(r)$

non-eg

- $\mathbb{R}^2 \setminus \{(0,0)\}$

eg

- disc $D_{\vec{r_0}}(r)$
- interior of a triangle
- interior of a convex polygon
- a half plane

  eg:

  $\{(x,y) \mid x > 0\}$

  $\{(x,y) \mid y < 2\}$

  $\{(x,y) \mid ax + by < c\}$
- $\{P \in \mathbb{R}^2 \mid a < \theta < b\}$
- quadrant

non-eg

- $\mathbb{R}^2 \setminus \{(0,0)\}$
- $\mathbb{R}^2 \setminus \{(7, -\sqrt{2})\}$
- $\mathbb{R}^2 \setminus \{(0,0), (1,2)\}$
- $\mathbb{R}^2 \setminus S$, for S a nonempty finite set of points
- $\mathbb{R}^2 \setminus S$, S any bounded nonempty closed subset

  eg: $S = \overline{D_{\vec{r_0}}(1)}$ ← a closed disc of radius 1
- any nonempty disc with a finite nonzero number of points removed

  eg: $D_{(3,4)}(2) \setminus \{(3,5), (4,3)\}$

Prop If S is a bounded nonempty closed subset of $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus S$ is not simply connected.

Proof Since S is bounded, then

$S \subseteq D_{(0,0)}(r)$ for sufficiently large r ($r >> 0$)

$\Rightarrow S \cap C_{(0,0)}(r) = \emptyset$

$\Rightarrow C_{(0,0)}(r) \subseteq \mathbb{R}^2 \setminus S$

but since S is nonempty, can find some $P \in S$

$\Rightarrow P \in D_{(0,0)}(r) = \mathrm{int}(C_{(0,0)}(r))$

$\Rightarrow \mathrm{int}(C_{(0,0)}(r)) \not\subseteq \mathbb{R}^2 \setminus S$

$\mathbb{R}^2 \setminus S$ is NOT simply connected

Thm Suppose $\vec{F}$ is a vector field on a simply-connected region R and $\mathrm{curl}(\vec{F}) = 0$. Then $\vec{F}$ is conservative on R.

Proof We will show $\vec{F}$ is conservative by showing $\int_C \vec{F} \cdot d\vec{r} = 0$

$\forall$ closed curve $C \subseteq R$.

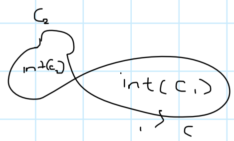
If C is simple closed then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\mathrm{int}(C)} \mathrm{curl}(\vec{F})\,dx\,dy$$

$$= \int_{\mathrm{int}(C)} 0\,dx\,dy$$

$= 0$

In general, suppose C crosses itself, eg:



divide C into 2 simple closed curves C_1 & C_2

Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot dr$$

because $C = C_1 - C_2$

minus bc CCW around C_1 goes CW around C_2

Now

Since the integrals over closed simple curves are 0, so is $\vec{F} \cdot d\vec{r}$

QED

eg Let $\vec{F}(x, y) = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$

Notice

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$\Rightarrow \text{curl}(\vec{F}) = 0$

(except at $(x, y) = (0, 0)$ where $\vec{F}$ is undefined)

$\vec{F}$ is a vector field on $\mathbb{R}^2 \setminus \{(0,0)\}$ but $\mathbb{R}^2 \setminus \{(0,0)\}$ isn't simply connected

Surface Integrals

Saturday, April 17, 2021 9:01 AM

$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

$\text{curl}(\vec{F}) = 0$ where $\vec{F}$ is defined (ie, on $\mathbb{R}^2 \setminus \{0,0\}$)

"$\vec{F}$ is irrotational" (ie, curl is 0)

conservative $\Rightarrow$ irrotational

irrotational $\Rightarrow$ conservative on a simply-connected domain

But $\mathbb{R}^2 \setminus \{(0,0)\}$ is not SC

Fact every irrotational vector field is locally conservative

ie, say $\vec{F}$ is irrotational on a domain D (ie, open in $\mathbb{R}^2$)

now $\vec{F}$ might not have a potential on D, but $\forall P \in D$, $\exists$ an open neighborhood containing P and contained in D on which $\vec{F}$ is conservative

ie, $\exists \varepsilon > 0$ s.t. $\vec{F}$ has a potential on $D_P(\varepsilon)$

but there might not be a single potential defined on all of D

θ is not a well-defined continuous fcn of $\mathbb{R}^2 \setminus \{(0,0)\}$

eg $\theta((1,1)) = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \quad \frac{\pi}{4} + 2\pi k$ for $k \in \mathbb{Z}$

Usually we choose $\theta \in [0, 2\pi)$

$\Rightarrow$ θ not continuous on positive x-axis

bc $\lim_{\varepsilon \to 0^+} \theta((1,\varepsilon)) = 0$

$\lim_{\varepsilon \to 0^-} \theta((1,\varepsilon)) = 2\pi$

In a sense $\theta((1,0)) = 0 \overset{?}{=} 2\pi$ ("multivalued fcn")

Recall FTC for line integrals

If $\vec{F} = \nabla F$, C is a curve from P to Q, then $\int_C \vec{F} \cdot d\vec{r} = F(Q) - F(P)$

Now take $F = \theta$, $\vec{F}$ as above

$$\int_{C_{(0,0)}(1)} \vec{F} \cdot d\vec{r} = \underset{2\pi}{\underset{\shortparallel}{F(1,0)}} - \underset{0}{\underset{\shortparallel}{F(1,0)}}$$

$$= 2\pi$$

Idea when you go around in a circle, you end up somewhere different (eg $\to$ a parking garage)

Notice θ is defined and continuous locally on $\mathbb{R}^2 \setminus \{(0,0)\}$

different (eg → a parking garage)

Notice θ is defined and continuous locally on $\mathbb{R}^2 \setminus \{(0,0)\}$

eg Define $\theta \in [-\pi, \pi)$

Then θ is cont on the positive x-axis but not on the negative x-axis

Winding #

(ie, how to determine whether $P \in \text{int}(C)$.)

Let C be a simple closed curve.

Let $P \in \mathbb{R}^2 \setminus C$. $P := (x_0, y_0)$

Consider

$$\int_C \vec{F}_P \cdot d\vec{r}$$

for $\vec{F} = \dfrac{(y-y_0)}{(x-x_0)^2 + (y-y_0)^2}\hat{i} + \dfrac{x-x_0}{(x-x_0)^2 + (y-y_0)^2}\hat{j}$

Then this integral is 2π if $P \in \text{int}(C)$

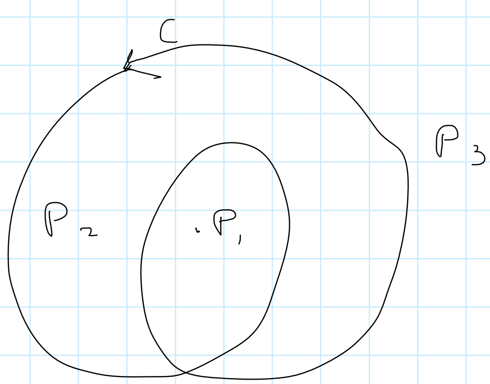
0 $P \in \text{ext}(C)$

In fact

$\dfrac{\int_C \vec{F}_P \cdot d\vec{r}}{2\pi}$ is the # of times that C goes around P.

↳ true even if C not simple closed

eg



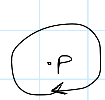
winding # of C around

$P_1 = 2$

$P_2 = 1$

$P_3 = 0$

eg



winding # around P is -1

Note given P, and loops C_1 and C_2 with same endpoint, then winding # of $C_1 + C_2$ around P is winding # of C_1 + winding # of C_2

Green's thm for multiply connected regions

$$\int_{\text{red region}} \text{curl } \vec{F} \, dx dy = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r}$$

critical C_3 is CW, C_1 & C_2 CCW

If curl $\vec{F} = 1$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\text{int}(C)} 1 \, dA = \text{area}(\text{int}(C))$$

Surface Integrals

Recall: 2 types of line integrals

① line integral of a scalar fcn (eg on $\mathbb{R}^2$)

For f def'd on a domain containing a curve C, we can take

$$\int_C f \, ds \qquad ds = \text{arclength}$$

Note ds always positive, and reversing the orientation of C doesn't change $\int_C f \, ds$

② line integral of a vector fcn.

For $\vec{F}$ def'd on a domain containing C, we take

$$\int_C \vec{F} \cdot d\vec{r}$$

Q/ Why dot product?

A/ Bc we have 2 vectors

$\vec{F}(x_i, y_i)$ and $\Delta x_i \hat{i} + \Delta y_i \hat{j} = \Delta \vec{r}$

and we want a scalar ⇒ dot prod

$\vec{F}(x_i, y_i)$ and $\Delta x_i \hat{i} + \Delta y_i \hat{j} = \Delta \vec{r}$

and we want a scalar $\Rightarrow$ dot prod!

<u>Note</u> $d\vec{r}$ is a vector and reversing the orientation of C negates the integral

<u>Recall</u> defined using a Riemann sum, but computed using a parameterization.

<u>Now</u> surfaces

Let S be a bounded surface in $\mathbb{R}^3$.

ie, suppose we have fcn

$\vec{r}(s,t) = (x(s,t), y(s,t), z(s,z))$

$\vec{r} : D \to \mathbb{R}^3$

$D \subseteq \mathbb{R}^2$ take $D = [a,b] \times [c,d]$

<u>Assume</u> x, y, z have continuous partials $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$

Then the subset of $\mathbb{R}^3$ traced out by $\vec{r}(s,t)$ for $(s,t) \in [a,b] \times [c,d] \subseteq \mathbb{R}^2$ is the kind of surface we care about.

<u>1 for surface</u> Let f be a scalar fcn defined on some domain in $\mathbb{R}^3$ containing a surface S.

will define $\int_S f \, dA$ $\quad A = \text{Area}$

<u>Riemann sum.</u> Break S into N little surfaces S_i for $i = 1, 2, \ldots, N$

- Choose $(x_i, y_i, z_i) \in S_i \ \forall_i$
- Then consider $\sum_{i=1}^{N} f(x_i, y_i, z_i) \,\text{area}(S_i)$
- define $\int_S f \, dA$ to be $\lim_{\text{mesh} \to 0}$ where $\text{mesh} = \max_i (\text{diam}(S_i))$

<u>To compute</u>

- Convert dA to $ds\,dt$
- Given a little rectangle w/ sides Δs & Δt it maps via $\vec{r}$ to a little parallelogram in S with sides $\frac{\partial \vec{r}}{\partial s} \Delta s$ and $\frac{\partial \vec{r}}{\partial t} \Delta t$

<u>Area of a parallelogram:</u>

$$\left\| \frac{\partial \vec{r}}{\partial s} \Delta s \times \frac{\partial \vec{r}}{\partial t} \Delta t \right\| = \Delta A$$

$$= \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| \Delta s \Delta t$$

$$\Rightarrow \int_S f\,dA = \iint_{[a,b]\times[c,d]} f \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| ds\,dt$$

Q/ What is $\frac{\partial \vec{r}}{\partial s}$?

A/ A VVF of s, t that outputs vectors in $\mathbb{R}^3$

② For surfaces

Suppose we have $\vec{F}(x, y, z) = P\hat{i} + Q\hat{j} + R\hat{k}$

Will consider

$$\int \underbrace{\left(\vec{F}, \frac{\partial \vec{r}}{\partial s}, \frac{\partial \vec{r}}{\partial t} \right)}_{\substack{\text{need this to} \\ \text{be a scalar} \\ \hookrightarrow \text{det?}}} ds\,dt$$

$$\begin{vmatrix} P & Q & R \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix}$$

In book consider $\vec{F} \cdot \hat{n}$

$\hookrightarrow$ this is the same bc $\vec{n} = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$

and $\det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1 \cdot (v_2 \times v_3)$

Riemann sum break S into pieces $S_1, \dots, S_N$

$$\sum_{i=1}^{N} f(x_i, y_i, z_i) \cdot \text{area}(S_i) \quad \text{s.t.} \quad (x_i, y_i, z_i) \in S_i$$

via parameterization $(x(s,t), y(s,t), z(s,t))$ for $(s,t) \in [a,b] \times [c,d]$

$$\iint f \, dA = \int_c^d \int_a^b f(x(s,t), y, z) \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| ds\, dt$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\vec{k} = (x, y, z)$$

Integral of a vector fcn over a surface

$$\iint_S \vec{F} \, d(\text{something}) = \int_c^d \int_a^b \det \begin{vmatrix} \vec{F} \\ \partial \vec{r}/\partial s \\ \partial \vec{r}/\partial t \end{vmatrix} ds\, dt$$

$$\det \begin{vmatrix} \vec{F} \\ \partial \vec{r}/\partial s \\ \partial \vec{r}/\partial t \end{vmatrix} = \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right)$$

$$= \left(\vec{F} \cdot \left[\frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|} \right] \right) \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|$$

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|}$$

← unit vector $\perp$ to S

unit normal vector

$\Rightarrow \iint_S (\vec{F} \cdot \vec{n}) \, dA$ is how we take an integral of a vector fcn

Notice if $\vec{F}$ always $\perp$ to S, then $\vec{F} \cdot \vec{n} = \|\vec{F}\|$

↳ in this case, surface integral of $\vec{F}$ is just $\iint \|\vec{F}\| \, dA$

What is orientation?

For line integrals, parameterization determines orientation

Idea: direction is from small t to larger t.

$(x(t), y(t)) \quad a \le t \le b$

reverse orientation $\quad (x(b+a-t), y(b+a-t))$

reverses orientation

In computing line integral, $x'(t)$, $y'(t)$ get negated.

For surface integrals, we have the factor

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}.$$

$$= -\left(\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial s}\right)$$

↳ To switch orientation of S, switch s & t

→ this should negate normal vector.

→ why?

RHR (right hand rule)

algebraic explanation

matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ switches the 2 coordinates and has det -1

$\Rightarrow$ this matrix reverses orientation

e.g. of orientations on surfaces

Sphere

↳ inward & outward pointing

Given a parameterization, how to know the way it's pointing?

$$S = \{x^2 + y^2 + z^2 = 1\}$$

~~For $0 \le s, t \le \frac{1}{2}$~~ :

$$(x, y, z) = (s, t, -\sqrt{1 - s^2 - t^2})$$

intuitively, (s,t) in same direction as (x, t)

$\Rightarrow$ RHR say $\vec{n}$ points up $\Rightarrow$ inward

(x, t)

$\Rightarrow$ RHR say $\vec{n}$ points up $\Rightarrow$ inward

algebraically

$$\frac{\partial \vec{r}}{\partial s} = \left(1,\ 0,\ -\frac{1}{2}(1-s^2-t^2)^{-\frac{1}{2}}(-2s)\right)$$

$$= \left(1,\ 0,\ \frac{s}{\sqrt{1-s^2-t^2}}\right)$$

$$\frac{\partial \vec{r}}{\partial t} = \left(0, 1, \frac{t}{\sqrt{1-s^2-t^2}}\right)$$

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \left(\frac{-s}{\sqrt{1-s^2-t^2}}, \frac{-t}{\sqrt{1-s^2-t^2}}, 1\right)$$

this points up from bottom of sphere $\Rightarrow$ inward

Plane (eg yz plane)

$\hookrightarrow$ 2 possible orientations are $+x$ direction ? $-x$ direction

ex: $(x, y, z) = (0, s, t)$

$$\frac{\partial \vec{r}}{\partial s} = (0, 1, 0) \qquad \frac{\partial \vec{r}}{\partial t} = (0, 0, 1)$$

and $(0,1,0) \times (0,0,1) = (1,0,0)$

$\Rightarrow$ positive x orientation

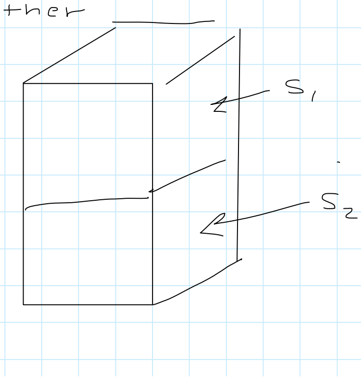
Green's Thm computes line integral of a vector field along a closed curve

Divergence Thm computes the surface of a vector field along a $\underbrace{\text{closed surface}}$

eg: sphere, cube, tetrahedron

$\hookrightarrow$ the boundary of a bounded solid in $\mathbb{R}^3$

Idea suppose we stack 2 cubes on top of each other



S_3 is boundary of rectangular prism formed by these cubes.

give all closed surfaces the outward orientation

$\Rightarrow$ on common face btwn 2 cubes, you have opposite orientation

$\Rightarrow$ cancellation of surface integrals

$$\Rightarrow \iint_{S_1} \vec{F}\cdot\vec{n}\, dA + \iint_{S_2} \vec{F}\cdot\vec{n}\, dA = \iint_{S_3} \vec{F}\cdot\vec{n}\, dA$$

In general, if we have a closed surface S s.t.

$S = \partial V$

"boundary of"

and we break up V into

$V = V_1 \cup V_2 \cup \dots \cup V_N$

Let $S_i = \partial V_i$, then:

$$\iint_S \vec{F}\cdot\vec{n}\, dA = \sum_{i=1}^{N} \iint_{S_i} \vec{F}\cdot\vec{n}\, dA$$

by the same cancellation idea as with stacked cubes

Idea of Div Thm

compute $\iint_S \vec{F}\cdot\vec{n}\, dA = \iint_S \vec{F}\cdot d\vec{\sigma}$

by breaking V into little pieces & adding them up then approximate the little pieces using derivative approximations for $\vec{F}$.

As the pieces get smaller, the approximation gets better & the sum becomes an integral

Consider a little piece V_i. Say it's a cube with verticies (x_i, y_i, z_i) and $(x_i + \Delta x, y_i, z_i)$ and $(x_i, y_i + \Delta y, z_i)$ and $(x_i, y_i, z_i + \Delta z)$ s.t. $\Delta x = \Delta y = \Delta z$ ($\Rightarrow$ cube)

This cube has 6 faces, which are divided into 3 pairs of opposite corresponding to 3 coord direction.

e.g. consider the opposite faces in the x-dir

Face 1 (x_i, y_i, z_i), $(x_i, y_i + \Delta y, z_i)$, $(x_i, y_i, z_i + \Delta z)$

$\hookrightarrow$ $-x$ orientation

Face 2 same but shifted in x-direction by Δx

$\hookrightarrow$ $+x$ orientation

b/c normal vector is on x-axis, we care only about the x-coord (aka $\vec{i}$-coord) of $\vec{F}$ for this pair of faces bc normal vector is on x-axis, we care only about the x-coord (aka $\hat{i}$-coord) of $\vec{F}$ for this pair of faces

say $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\Rightarrow \iint_{\text{face } 1}$$

$$\Longrightarrow \iint_{\text{face } 1} \vec{F}\cdot d\vec{\sigma} \approx P(x_i, y_i, z_i)\Delta y \Delta z$$

$$\Longrightarrow \iint_{\text{face } 2} \vec{F}\cdot d\vec{\sigma} \approx P(x_i + \Delta x, y_i, z_i)\Delta y \Delta z$$

$$\Longrightarrow \iint_{\text{face } 1} \vec{F}\cdot d\vec{\sigma} + \iint_{\text{face } 2} \vec{F}\cdot d\vec{\sigma}$$

$$= P(x_i + \Delta x, y_i, z_i) - P$$

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$$\Longrightarrow \iint_{\text{face } 1} \vec{F}\cdot d\vec{\sigma} + \iint_{\text{face } 2} \vec{F}\cdot d\vec{\sigma}$$

$$= \left[P(x_i+\Delta x, y_i, z_i) - P(x_i, y_i, z_i)\right]\Delta y \Delta z$$

$$\approx \left[\frac{\partial P}{\partial x}(x_i, y_i, z_i)\Delta x\right]\Delta y \Delta z.$$

Derivation/proof of Divergence Thm

Recall for Green's Thm

Say we have a simple closed curve bounding a region A. $C = \partial A$

Thm

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F} \, dA$$

Cancellation along overlapping edges

$\Rightarrow$ If $R = R_1 \cup R_2 \cup \cdots \cup R_N$ and let $C_i = \partial R_i$ then

$$\int_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^{N} \int_{C_i} \vec{F} \cdot d\vec{r}$$

$\Rightarrow$ Green's Theorem

Cancellation along overlapping faces for surface integrals

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{\sigma} = \sum_{i=1}^{N} \iint_{S_i} \vec{F} \cdot d\vec{\sigma}$$

$S = \partial R$ is a closed surface

R open connected region in $\mathbb{R}^3$

$R = R_1 \cup R_2 \cup \cdots \cup R_N$

and $S_i = \partial R_i$

$R = R_1 \cup R_2 \cup \cdots \cup R_N$
and $S_i = \partial R_i$

this fact is what tells you that

$$\iint_S \vec{F} \cdot d\vec{a} \quad \text{can be}$$

expressed as a triple integral.

<u>Last time</u>: let R_i be a little cube. Then $S_i = \partial R_i$ has 6 faces in 3 pairs (corresponding to x, y, z)

↳ we showed that the sum of the integrals over the pair of faces in the x direction, ie, $\perp$ to x-axis is

$$\approx \frac{\partial P}{\partial x} \Delta x \Delta y \Delta z$$

$$= \frac{\partial P}{\partial x} \operatorname{vol}(R_i)$$

where $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

↳ Similarly, can show that the sum of the integrals over the faces in the y-direction is

$$\approx \frac{\partial Q}{\partial y} \operatorname{vol}(R_i)$$

↳ in z-dir:

$$\frac{\partial R}{\partial z} \operatorname{vol}(R_i)$$

$$\Rightarrow \iint_{S_i} \vec{F} \cdot d\vec{\sigma} \approx \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{vol}(R_i)$$

gets better as mesh$(\{R_i\}) \to 0$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{\sigma} = \sum_{i=1}^{N} \iint_{S_i} \vec{F} \cdot d\vec{\sigma}$$

$$\approx \sum_{i=1}^{N} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{vol}(R_i)$$

Now take limit as mesh $\to 0$
(to simplify, think of it as $N \to \infty$)
and get

$$\iint_{\partial R} \vec{F} \cdot d\vec{\sigma} = \iiint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

divergence thm

For a vector field $\vec{F}$ in $\mathbb{R}^3$:

$$\mathrm{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

<u>Recall</u> for a fcn f on an inverval $[a,b]$,

$$\frac{1}{\mathrm{length}([a,b])} \int_a^b f(x)\,dx$$

is the avg value of f on $[a,b]$

<u>Similarly</u> for f defined on a region R in $\mathbb{R}^2$,

$$\frac{1}{\mathrm{area}(R)} \iint f\,dA = \text{average of } f \text{ on the region } R$$

$$\frac{1}{\text{area}(R)} \iint_R F\, dA = \text{average of } F \text{ on the region } R$$

If R is a region on $\mathbb{R}^3$ and F defined on R,

$$\frac{1}{\text{vol}(R)} \iiint F\, dV = \text{avg of } F \text{ on } R$$

$$\frac{\iiint_{\partial R} \vec{F} \cdot d\vec{\sigma}}{\text{vol}(R)} = \text{avg of div } \vec{F} \text{ on } R$$

Suppose div $\vec{F}$ is const

$\underline{\text{eg}}$ $\vec{F}$ is linear

$$\vec{F} = xy\hat{i} - \frac{y^2}{2}\hat{j} + (xy + 3z)\hat{k}$$

$$\text{div } \vec{F} = 3$$

then

$$\iiint_{\partial R} \vec{F} \cdot d\vec{\sigma} = \text{vo}(R) \cdot C$$

$$\Rightarrow \text{vol}(R) = \frac{\iiint_{\partial R} \left(xy\hat{i} - \frac{y^2}{2}\hat{j} + (xy + 3z)\hat{k}\right) \cdot d\vec{\sigma}}{3}$$

$\underline{\text{Note}}$: If R really small and div $\vec{F}$ is continuous, then div $\vec{F} \approx$ constant

Try $R = \left\{ \vec{r} \in \mathbb{R}^3 \;\middle|\; \|\vec{r} - \vec{r}_0\| < r \right\}$

= sphere of radius r around $\vec{r}_0$

$$\lim_{r \to 0} \frac{\iint_{\partial R} \vec{F} \cdot d\vec{\sigma}}{\mathrm{vol}(R)} = \operatorname{div} \vec{F}(\vec{r}_0)$$

$$= \frac{4}{3}\pi r^3$$

Intuitive / Physical Description of $\iint_S \vec{F} \cdot d\vec{\sigma}$

Recall $\vec{F} \cdot d\vec{\sigma} = \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds\,dt$

$$= \vec{F} \cdot \vec{n}\, dA$$

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|}$$

picks out the component of $\vec{F}$ that is perpendicular to S at the given point.

Suppose $\vec{F}$ represents velocity of air at a given point

Suppose there's an open door and let S be the surface bounded by the doorframe

Then $\iint_S \vec{F} \cdot d\vec{\sigma}$ is the rate at which air is transferred btwn the 2 rooms

Say the door is btwn Room A and Room B, then an orientation for S is either

$A \to B$ or
$B \to A$

If we fix orientation to be $A \to B$ then if $\iint_S \vec{F} \cdot d\vec{\sigma}$, it means

more air is flowing from $A \to B$ than $B \to A$

If negative, B is losing air, A is gaining air.

Why dot with $\vec{n}$?

If we want to know how much air is flowing from $A \to B$ (or vice versa)

$\to$ Latin flux $\to$ flux integral

We care only about $\vec{F}$ that is $\perp$ to the door

e.g. if the air is flowing in a way $\parallel$ to the door, it shouldn't move between the rooms

Stokes Thm

(basically Green's thm in 3 dim)

(basically Green's thm in 3 dim)

↳ ie, express a line integral in terms of a surface integral $\iint$

Green's thm

$\vec{F} = P\,dx + Q\,dy$

in $\mathbb{R}^2$ viewed as xy-plane in $\mathbb{R}^3$

say $C = \partial R$ for R a 2-D region in $\mathbb{R}^2$

then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F} \cdot dA$$

$$\stackrel{?}{=} \iint_S \vec{g} \cdot \vec{n}\, dA$$

then need: $\text{curl}\, \vec{F} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

$= \vec{g} \cdot \vec{n}$

S should just just be R (a subset of $\mathbb{R}^2$)

Bc S is in the xy plane, $\vec{n} = \hat{k}$

↳ (up; NOT $-\hat{k}$ bc

C goes CCW (RHR)

$\Rightarrow \vec{k}$ - component of $\vec{g}$ should be

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Q/ What about other components of $\vec{g}$?

If we want to generalize to 3 dim, want

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\Rightarrow \vec{g} = (\ )\hat{i} + (\ )\hat{j} + \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}}_{\text{looks like } \vec{k} \text{ component of cross prod}}$$

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(P\hat{i} + Q\hat{j} + R\hat{k}\right)$$

= Vector curl in 3-dim

Stokes Theorem

Tuesday, April 27, 2021 11:14 AM

Green's Theorem

Let D be an open domain in the plane $\mathbb{R}^2$ (think of $\mathbb{R}^2$ as xy plane in $\mathbb{R}^3$)

Suppose $\vec{F}$ is a vector field defined on D

ie $\vec{F} = P\hat{i} + Q\hat{j} + 0\hat{k}$ (will generalize to $R\hat{k}$)

and C is a simple closed curve in D s.t.

$\text{int}(C) \subseteq D$

Green Thm (Classical Form)

$$\int_C \vec{F} \cdot d\vec{r} = \iint_{\text{int}(C)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Green's Thm (Rewritten)

Think of $\text{int}(C)$ as a surface in $\mathbb{R}^3$ with orientation upward (in z-dir.)

Let's consider a vector field $\vec{g}$ with $\vec{g} \cdot \vec{k} = \vec{k}$-component of $\vec{g}$.

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Notice: $C = \underbrace{\partial}_{\text{"boundary"}} (\text{int}(C))$

$$\int_{\partial(\text{int}(C))} \vec{F} \cdot d\vec{r} = \underbrace{\iint_{\text{int}(C)} \vec{g} \cdot d\vec{\sigma}}_{\text{surface integral}}$$

Note: For $\text{int}(C)$ viewed as a surface in $\mathbb{R}^3$, it's normal vector $\vec{n}$ is $\vec{k}$

$\Rightarrow \iint_{\text{int}(C)} \vec{g} \cdot d\vec{\sigma}$ depends only on $\vec{k}$ component of $\vec{g}$

Stokes Thm

Let D be a domain in $\mathbb{R}^3$.

Let Σ be a surface in D.

Suppose $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field defined on D, differentiable

Then for an appropriate vector field $\vec{g}$ (determined by $\vec{F}$) with $\vec{k} \cdot \vec{g} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

We have:

$$\int_{\partial(\Sigma)} \vec{F} \cdot d\vec{r} = \iint_{\Sigma} \vec{g} \cdot d\vec{\sigma}$$

with consistent orientations via RHR

Remark

the RHS depends on Σ, while LHS depends only on $\partial\Sigma$

~~eg~~ $\Sigma_1 =$ northern hemisphere of unit sphere

$\Sigma_2 =$ southern hemisphere of unit sphere

then $\partial\Sigma_1 = \partial\Sigma_2 =$ equator

therefore

$$\iint_{\Sigma_1} \vec{g} \cdot d\vec{\sigma} = \iint_{\Sigma_2} \vec{g} \cdot d\vec{\sigma}$$

caveat may be $\pm$ depending on orientations

For (1)

C_2

caveat may be $\pm$ depending on orientations
for (+), need both upward or both downward

This is similar to the statement that if C is a curve from P to Q, and $\vec{g} = \nabla F$, then

$$\int_C \vec{g} \cdot d\vec{r} = F(Q) - F(P)$$

$\rightarrow$ so the LHS depends only on the endpoints of C, (i.e. $\partial(C)$) and not a particular path btwn them

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What is $\vec{g}$ in terms of $\vec{F}$?

Recall

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$\vec{k} \cdot \vec{g} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Idea pretend we have a "vector":

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$= \vec{g}$

this is the $\vec{g}$ in terms of $\vec{F}$, making Stokes Thm true

Recall for $\vec{F} = P\hat{i} + Q\hat{j}$ in the plane, we defined

scalar fcn

$$\operatorname{curl} \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \longleftarrow \hat{k}\text{-component of curl } \vec{F} \text{ for } \vec{F} \text{ in } xy \text{ plane}$$

In fact the usual defn of curl is for 3-dim vector fields and produces another vector field, it is

$$\nabla \times \vec{F} = \operatorname{curl} \vec{F}$$

Statement of Stokes

For a vector field $\vec{F}$, surface Σ, all in some domain $\mathbb{R}^3$,

$$\int_{\partial\Sigma} \vec{F} \cdot d\vec{r} = \iint_{\Sigma} \operatorname{curl} \vec{F} \cdot d\vec{\sigma}$$

Let Σ = disc of radius r around $P \in \mathbb{R}^3$, parallel to xy-plane

eg, if $P = (2,3,4)$, then this disc lies in the plane $z=4$

Let $P = (x_0, y_0, z_0)$

$$\int_{\partial\Sigma} \vec{F} \cdot d\vec{r} = \iint_{\Sigma} \operatorname{curl} \vec{F} \cdot d\vec{\sigma}$$

$$= \iint_{\Sigma} (\operatorname{curl} \vec{F} \cdot \vec{k})\, dA$$

$$\approx (\operatorname{curl} \vec{F}(P) \cdot \vec{k})\operatorname{area}(\Sigma)$$

$\hookrightarrow$ As $r \to 0$, the $\approx$ gets better

$$\Rightarrow \operatorname{curl} \vec{F}(P) \cdot \vec{k} = \lim_{r \to 0} \frac{\int_{\partial\Sigma} \vec{F} \cdot d\vec{r}}{\operatorname{area}(\Sigma)}$$

Recall what about Σ gives the $\underbrace{\hat{k}\text{-component}}_{\text{bc } \Sigma \text{ is parallel to } xy\text{-plane}}$ of curl $\underbrace{\text{at } P}_{\text{bc } \Sigma \text{ is a little disc around } P \Rightarrow \partial\Sigma \text{ is a circle around } P}$?

Rewrite

Let $C_r^{xy}(P)$ = circle of radius r and center P lying in a plane parallel to xy-plane

then for any continuously diff'able vector field $\vec{F}$

$$\hat{k}\cdot \operatorname{curl}\vec{F}(P) = \lim_{r\to 0} \frac{\int_{C_r^{xy}(P)} \vec{F}\cdot d\vec{r}}{\pi r^2}$$

"Curl as circulation"

Simil let $C_r^{yz}(P)$ = circle of radius r and center P lying in a plane parallel to yz-plane

$P = (x_0, y_0, z_0)$, then

$$C_r^{yz}(P) = \left\{ (x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{matrix} x = x_0 \\ (y-y_0)^2 + (z-z_0)^2 = r^2 \end{matrix} \right\}$$

Then $$\hat{i}\cdot \operatorname{curl}\vec{F}(P) = \lim_{r\to 0} \frac{\int_{C_r^{yz}(P)} \vec{F}\cdot d\vec{r}}{\pi r^2}$$

Similary

$$\hat{j}\cdot \operatorname{curl}\vec{F}(P) = \lim_{r\to 0} \frac{\int_{C_r^{xz}(P)} \vec{F}\cdot d\vec{r}}{\pi r^2}$$

Theorem $\operatorname{curl}(\nabla F) = 0$

Proof 1

write $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ then $\operatorname{curl}(\nabla F)$ using the fact that mixed partials commute

Heuristic if we think of $\vec{\nabla}$ as a vector:

∇F is like $\vec{\nabla}$ times scalar F

$\Rightarrow \nabla F$ is "parallel" to $\vec{\nabla}$

$\Rightarrow \vec{\nabla}\times(\vec{\nabla} F) = 0$

Proof 2

path integral of ∇F is path-ind (bc conservative)

$\Rightarrow \int_C \nabla F\cdot d\vec{r} = 0$ if C a closed curve

eg, for $C = C_r^{xy}(P), C_r^{yz}(P), C_r^{xz}(P)$

$\Rightarrow$ each component of $\operatorname{curl}(\nabla F)$ at any point P in the domain of F is

$$\lim \frac{0}{\;} = 0$$

domain of F is

$$\lim_{r \to 0} \frac{0}{\pi r^2} = 0$$

$\Rightarrow \text{curl}(\nabla F) = 0$

grad : takes scalar fcn F to vector field $\vec{\nabla} F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$

curl : takes vector field $\vec{F}$ to vector field $\vec{\nabla} \times \vec{F} =$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} \right)$$

for $\vec{F} = (P, Q, R)$

Div : takes vector field $\vec{F} = (P, Q, R)$ to scalar fcn $\vec{\nabla} \cdot \vec{F} =$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Last time : curl(grad(F)) = 0

Proof 1 use mixed partials

Remark : makes sense b/c cross product of two parallel vectors is 0.